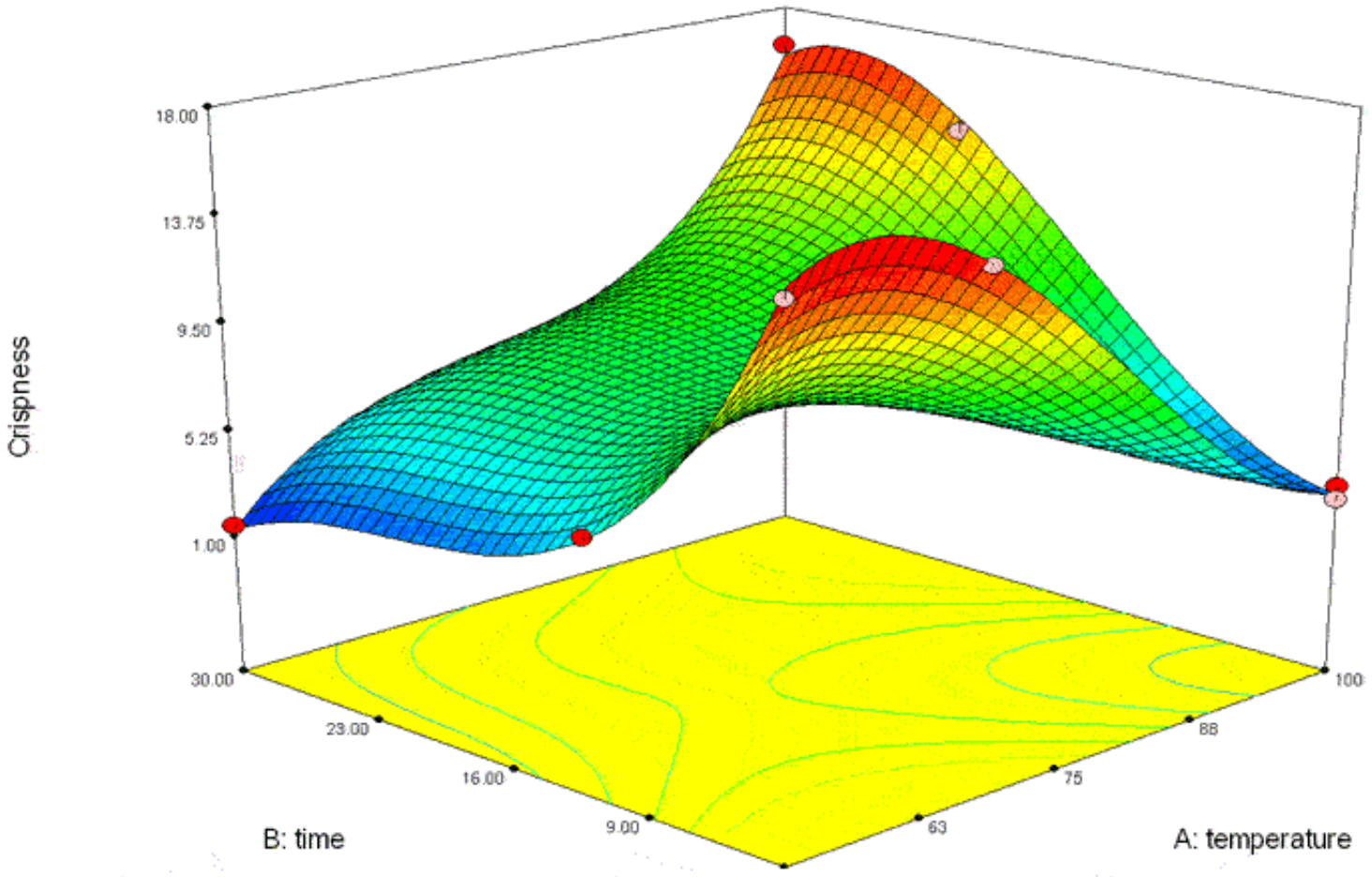
Design of Experiments (DoE)



Design of Experiments (DoE): the first developments

Statistical experimental design, or design of experiments (DoE) is the methodology referred to how to conduct and plan experiments to extract the maximum amount of information in the fewest number of runs.

The first example of DoE dates back to 1923, with a study on the response of different potato varieties to fertilizers, published by the famous English statistician Ronald Fisher.

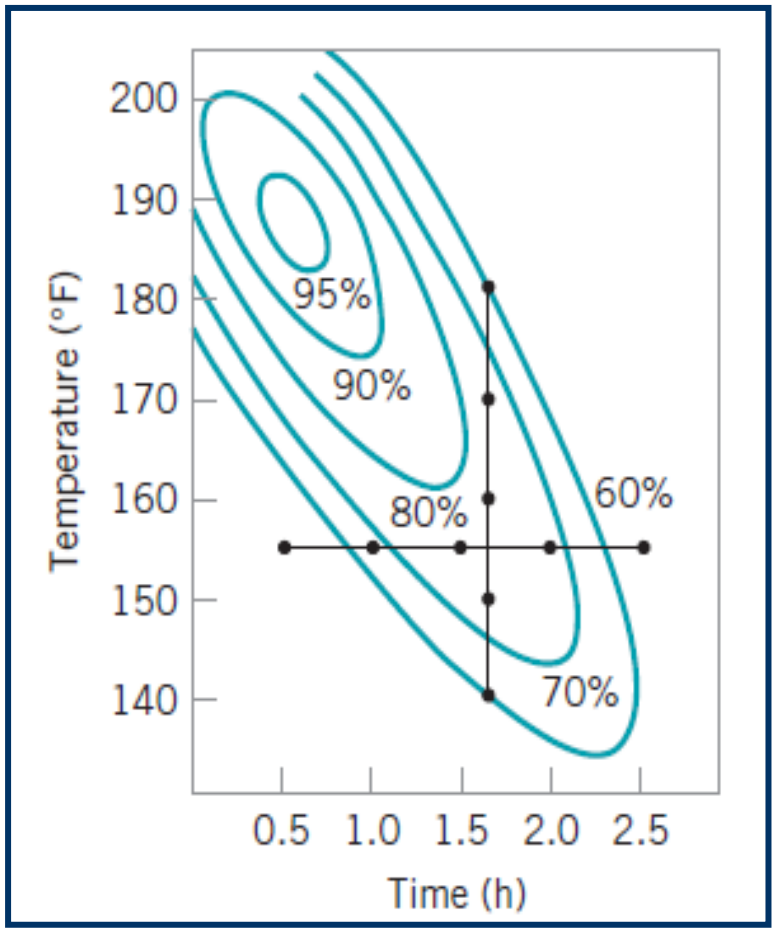
Fisher published the first book on the topic in 1935 (in the same book the nomenclature of hypothesis testing was also introduced).

During the 1950s the English statistician George E.P. Box published several papers on fundamental aspects of experimental design.

Problems with the «One variable at a time (OVAT)» approach

Although it is widely adopted in scientific research, the approach based on the consideration of the effect of one variable at a time (OVAT) can lead to poor results, eventually preventing the researcher from finding the really optimal conditions.

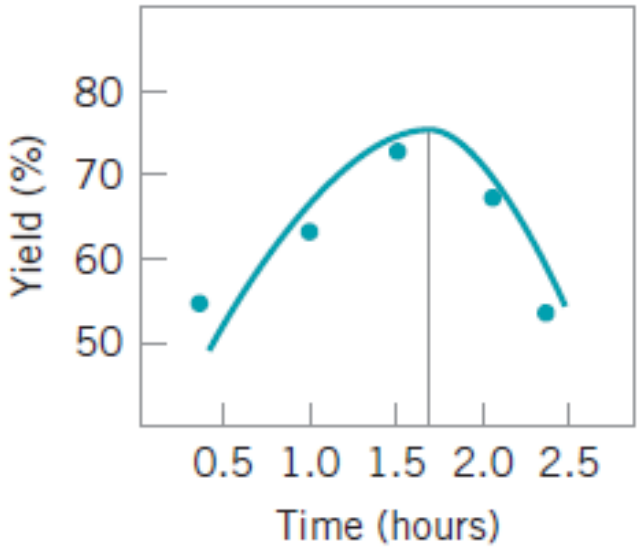
As an example, a contour plot of a reaction yield as a function of time and temperature is reported in the figure on the right:



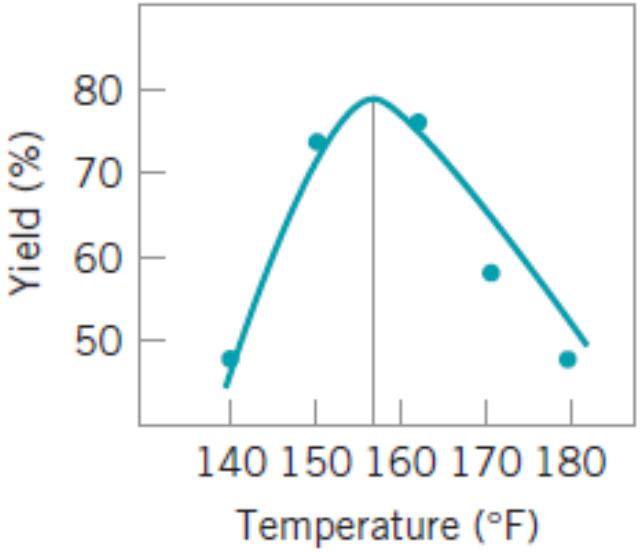
As apparent, the yield tends towards 100% when high temperatures and short times are adopted.

If temperature and time were explored separately, as indicated by the two black lines in the figure, the trends shown in the next slide would be observed.

Note: °C = (°F – 32) × 5/9



Yield vs reaction time at constant temperature (155°F).



Yield vs temperature at constant time (1.7 h).

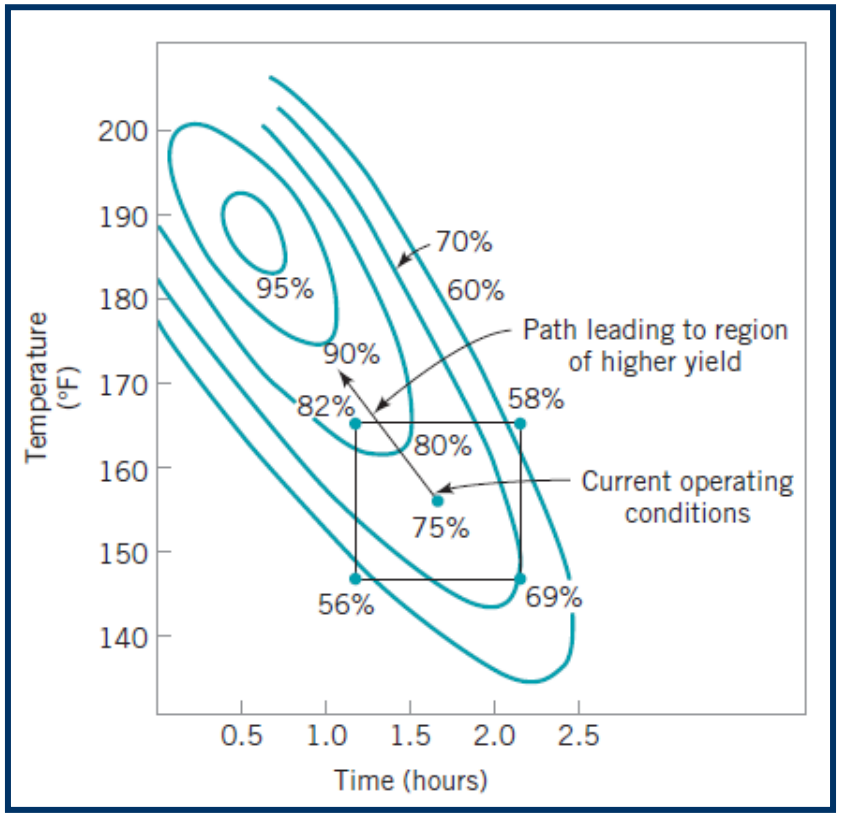
Since the appropriate combination of reaction temperature and time was not explored, the maximum yield achieved using the OVAT approach (lower than 80%) was much lower than the actual maximum yield.

To locate the real optimum, an experiment that varies time and temperature together (the only way to detect interactions) should be performed.

This type of experiment is called a factorial experiment.

The responses observed at the four corners of the square indicate that we should move in the general direction of increased temperature and decreased reaction time to increase yield.

Two approaches might be used:



Method of steepest ascent

A few additional runs could be performed in the direction found before, which would be sufficient to locate the region of maximum yield.

Response surface methodology

After reaching the region of the optimum a more elaborate experiment could be performed to obtain a very precise estimate of the optimal operating condition.

Comparison between DoE and OVAT approaches

- ✓ DoE considers the interactions among the variables, while the OVAT does not;
- ✓ DoE provides a global knowledge (in the whole experimental domain), while OVAT gives a local knowledge (only where the experiments have been performed);
- ✓ In each point of the experimental domain, the quality of the information obtained by the experimental design is higher than the information obtained by the OVAT;
- ✓ The number of experiments required by an experimental design is smaller than the number of experiments performed with an OVAT approach.

The most important aspect of DoE is that it provides a strict mathematical framework for changing all pertinent factors simultaneously and achieve this in a small number of experimental runs.

DoE basic terminology

Experimental domain	the experimental 'area' that is investigated, defined by the variation of the experimental variables.
Factors	experimental variables that can be changed independently of each other
Independent variables	same as factors
Levels	values assumed by factors
Continuous variables	independent variables that can be changed continuously
Discrete variables	independent variables that can be changed step-wise
Responses	the measured value of the results from experiments
Residual	the difference between the calculated and the experimental result

Steps for designing an experiment

1. Recognition of/statement of the problem: can be very helpful in terms of process understanding

2. Choice of factors and levels: process knowledge, based on practical experience and theoretical understanding, is required; in a first step the number of factor levels is usually kept low (*e.g.*, 2)

3. Selection of the response: it should correspond to a quantity providing useful information on the process under study; sometimes multiple responses are considered

4. Choice of experimental design: selection of sample size (number of replicates) and run order for experimental trials, eventual introduction of blocking or other randomization restrictions

5. Performing the experiment: at this stage great care must be put in experimental procedures; in fact, eventual errors would decrease significantly the experimental validity

6. Data analysis: statistical methods should be used in this step, to obtain objective results. Software packages are often used.

7. Conclusions and recommendations: draw practical conclusions about the results and recommend a course of action. Follow-up runs and confirmation testing should also be performed to validate the conclusions.

Empirical models

It is reasonable to assume that the outcome of an experiment is dependent on the experimental conditions. This means that response, y, can be described as a function based of the experimental variables x_i; a contribution due to experimental error, ε, has also to be considered. Typical multivariate models adopted are the following:

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \epsilon$$

first-degree model

k = number of factors

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum \sum \beta_{ij} x_i x_j + \epsilon$$

interaction model

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum \sum \beta_{ij} x_i x_j + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \epsilon$$

second-degree model

Once estimates of model coefficients are obtained, it is possible:

1) to establish a relationship, albeit approximate, between y and x_1, x_2, . . . , x_k, that can be used to predict response values for given settings of the control variables.

2) to determine, through hypothesis testing, the significance of the factors whose levels are represented by x_1, x_2, . . . , x_k

3) to determine the optimum settings of x_1, x_2, . . . , x_k, i.e., settings that result in the maximum (or minimum) response over a certain region of interest.

As shown in the following for models based on two variables, the difference between the actual response and the one reproduced by the estimated model corresponds to the residual:

$$y = b_0 + b_1 x_1 + b_2 x_2 + \text{residual}$$ ***first-degree (linear) model***

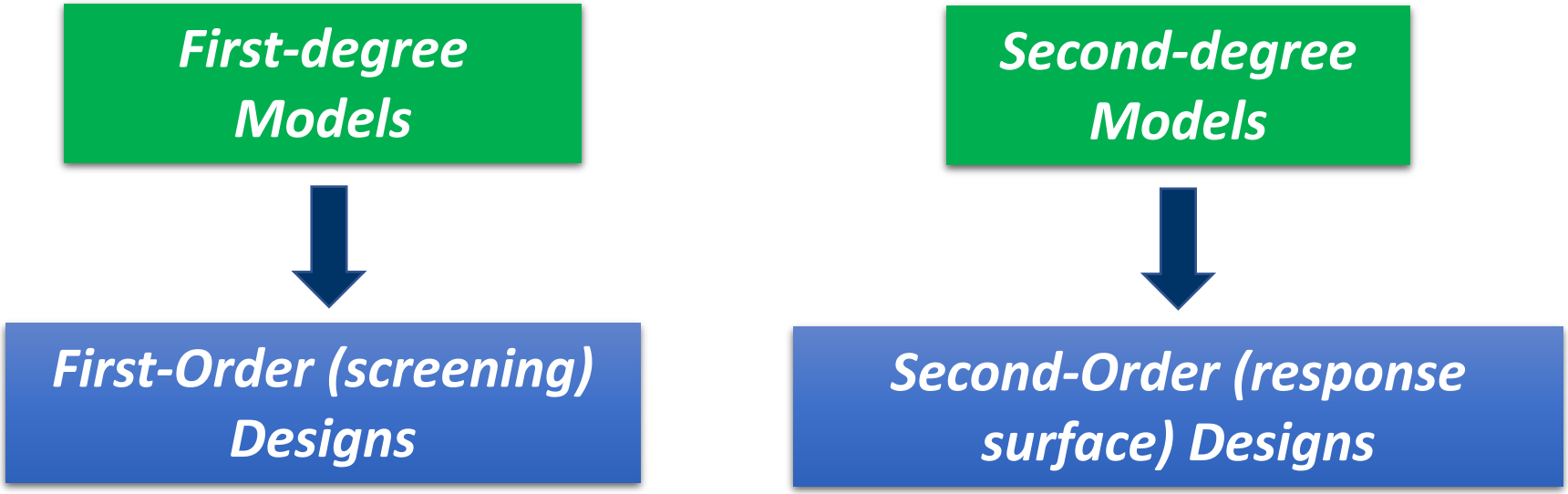
$$y = b_0 + b_1 x_1 + b_2 x_2 + b_{12} x_1 x_2 + \text{residual}$$ ***interaction model***

$$y = b_0 + b_1 x_1 + b_2 x_2 + b_{11} x_1^2 + b_{22} x_2^2 + b_{12} x_1 x_2 + \text{residual}$$ ***second-degree (quadratic) model***

The minimum number of experiments required in the three cases to estimate parameters is: 3 (linear model), 4 (interaction model), 6 (quadratic).

Designs related to First- and Second-Degree Models

The following classification can be made for experimental designs, based on their relationship with first- or second-degree models:



Screening designs

Screening designs provide simple models with information about dominating variables and ranges (relevant information is gained in only a few experiments).

In accordance with the Pareto's principle, the 20% of factors usually account for the 80% of information, thus information on dominating variables can be very useful.

First-degree or interaction models are usually adopted when screening designs are performed.

Actually, the main interest is in understanding if a factor does influence the response, not how it influences it.

Two-level factorial designs are some of the most common screening designs.

Factorial designs

The term “factorial design“ is thought to have been introduced for the first time in the statistical literature by Ronald Fisher in its 1935 book on DoE.

As a general definition, a factorial design consists of two or more factors, each with discrete possible values or "levels“.

If experimental units take on all possible combinations of these levels across all such factors the design is defined as “full factorial”.

Conversely, a “fractional factorial” design consists of a carefully chosen subset (fraction) of the experimental runs of a full factorial design.

The subset is chosen so as to exploit the sparsity-of-effects principle, i.e., to expose information about the most important features of the problem studied while using a fraction of the effort of a full factorial design in terms of experimental runs and resources.
In other words, it considers that many experiments in full factorial design are often redundant, giving little or no new information about the system.

Two-level factorial designs (2^k designs)

Two-level factorial designs consider two values for each factor as input.

First, a range is carefully chosen for each factor, to ensure feasibility, then two possible values for each factor, low and high, are selected.

Such levels are subsequently coded (normalized) so that their values range between -1 and +1.

The following formula is adopted for coding in the case of two level-factors:

$$\text{Coded level} = (\text{Uncoded level} - M)/H$$

where M is the average of high and low uncoded levels and H is the half-width of their interval.

Examples of full 2^k factorial designs

Two variables			Three variables				Four variables				
Exp. no.	Variables		Exp. no.	Variables			Exp. no.	Variables			
	x_1	x_2		x_1	x_2	x_3		x_1	x_2	x_3	x_4
1	−	−	1	−	−	−	1	−	−	−	−
2	+	−	2	+	−	−	2	+	−	−	−
3	−	+	3	−	+	−	3	−	+	−	−
4	+	+	4	+	+	−	4	+	+	−	−
			5	−	−	+	5	−	−	+	−
			6	+	−	+	6	+	−	+	−
			7	−	+	+	7	−	+	+	−
			8	+	+	+	8	+	+	+	−
							9	−	−	−	+
							10	+	−	−	+
							11	−	+	−	+
							12	+	+	−	+
							13	−	−	+	+
							14	+	−	+	+
							15	−	+	+	+
							16	+	+	+	+

+ and – signs represent high (+1) and low (-1) levels of coded factors (variables).

Notably, all variables are changed in a controlled way, to ensure that every experiment is a unique combination of levels. As apparent, the total number of experiments correspond to 2^k.

2^1 full factorial design

Suppose a reaction yield has been measured at two temperatures, resulting equal to 70 and 80% for temperatures equal to 50 and 100 °C, respectively.
The temperature needed to get 90% yield has to be estimated.

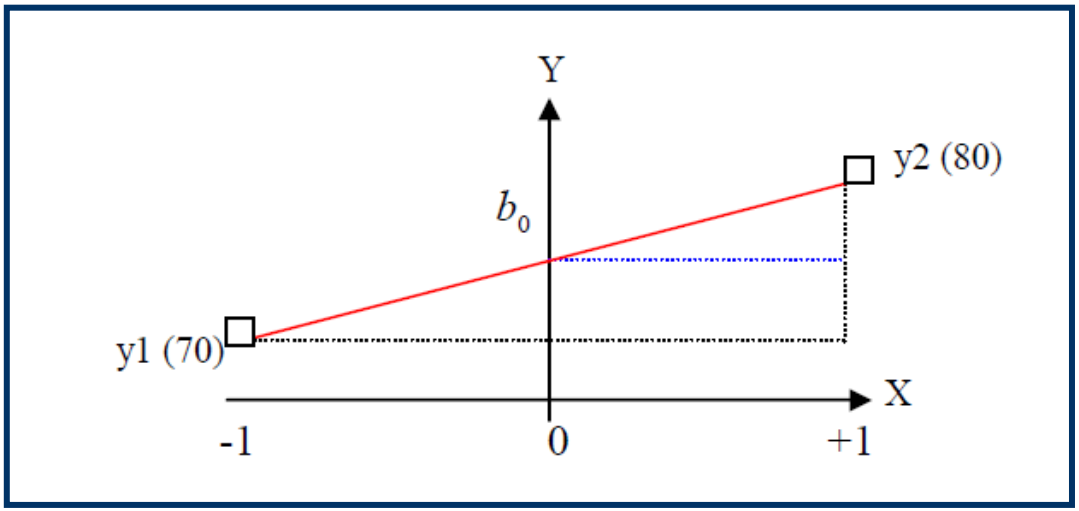
A linear model can be adopted:

$$y = b_0 + b_1 x_1 + \varepsilon$$

where x_1 represents the reaction temperature, after coding, and y the yield.
In this case 50 and 100°C temperatures are coded as -1 and +1, respectively.

Under these conditions:
b_0 = 75 % and b_1 = 5%

Thus a yield of 90% is obtained for
x_1 = (90-75)/5 = 3
i.e., for a temperature
(75 + 3 × 25°C) = 150°C.

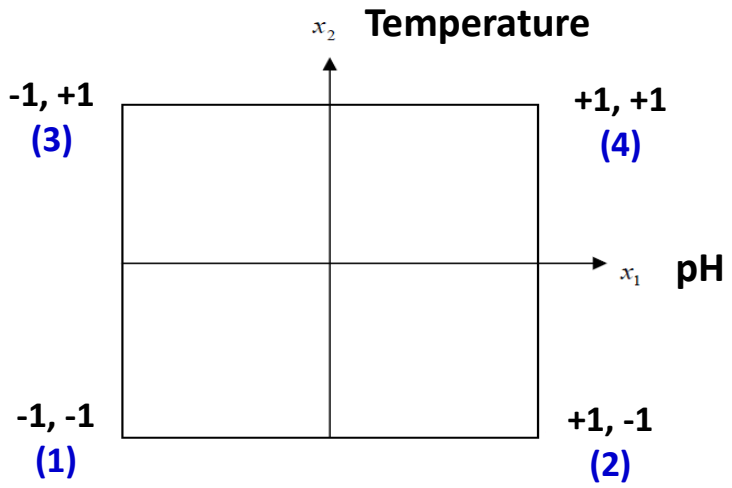


2^2 full factorial design

Let us consider a design involving pH (levels 3 and 7) and temperature (40 and 80°C).

The following experimental matrix, *i.e.*, a matrix describing all non redundant combinations of levels, and graphical representation of the experimental domain can be drawn:

Experiment	X1 pH	X2 temperature
1	-1	-1
2	+1	-1
3	-1	+1
4	+1	+1



Let us consider an interaction model:

$$y = b_0 + b_1 x_1 + b_2 x_2 + b_{12} x_1 x_2$$

A model matrix, reporting values by which parameters b_0, b_1, b_2 and b_{12} have to be multiplied, according to the selected experiment, can be drawn:

		pH	T	pHT	
Exp.	**Const.**	x_1	x_2	$x_1 x_2$	**y**
1	+1	-1	-1	+1	50
2	+1	+1	-1	-1	70
3	+1	-1	+1	-1	80
4	+1	+1	+1	+1	90

The response y obtained from each experiment is reported outside the matrix.

Interestingly, the effect of pH on the response depends on temperature, since for the same pH variation the response variation is higher (20 vs 10) at the lower temperature.
This outcome indicates the presence of an interaction between the two factors.

This conclusion is obviously confirmed by the fact that the response variation with temperature is higher (30 vs 20) at the lower pH.

Now, the average response variations due to pH and temperature variations are, respectively, 15 and 25.
Since they correspond to variations of 2 units of the coded factors (from -1 to + 1), it can then be inferred that the average response variations due to pH and temperature are, respectively, 7.5 and 12.5.

These values can be adopted as estimates of coefficients b_1 and b_2, *i.e.*, those related to pH and temperature, respectively, in the model.

The interaction effect is calculated according to one of the two following equations:

$[(X_1 \text{ effect at high } X_2)-(X_1 \text{ effect at low } X_2)]/2 = [(y_4-y_3) - (y_2-y_1)]/2 = (y_1-y_2-y_3+y_4)/2$

$[(X_2 \text{ effect at high } X_1)-(X_2 \text{ effect at low } X_1)]/2 = [(y_4-y_2) - (y_3-y_1)]/2 = (y_1-y_2-y_3+y_4)/2$

Since $(y_1-y_2-y_3+y_4)/2 = (50-70-80+90)/2 = -5$ and the effect needs to be divided by the range of each factor, i.e., +1 - (-1) = 2, the coefficient b_{12}, accounting for interaction effects, is equal to -5/2 = -2.5.

As far as the b_0 coefficient is concerned, it is easy to understand that it can be obtained by averaging the responses obtained for all the four experiments.

In fact, considering the coded values assumed by x_1, x_2 and x_1x_2 for the four different experiments, the following equations are obtained:

$$(y_1 + y_2 + y_3 + y_4)/4 = \{[b_0 - b_1 - b_2 + b_{12}] + [b_0 + b_1 - b_2 - b_{12}] + [b_0 - b_1 + b_2 - b_{12}] + [b_0 + b_1 + b_2 + b_{12}]\}/4 = 4\,b_0/4 = b_0$$

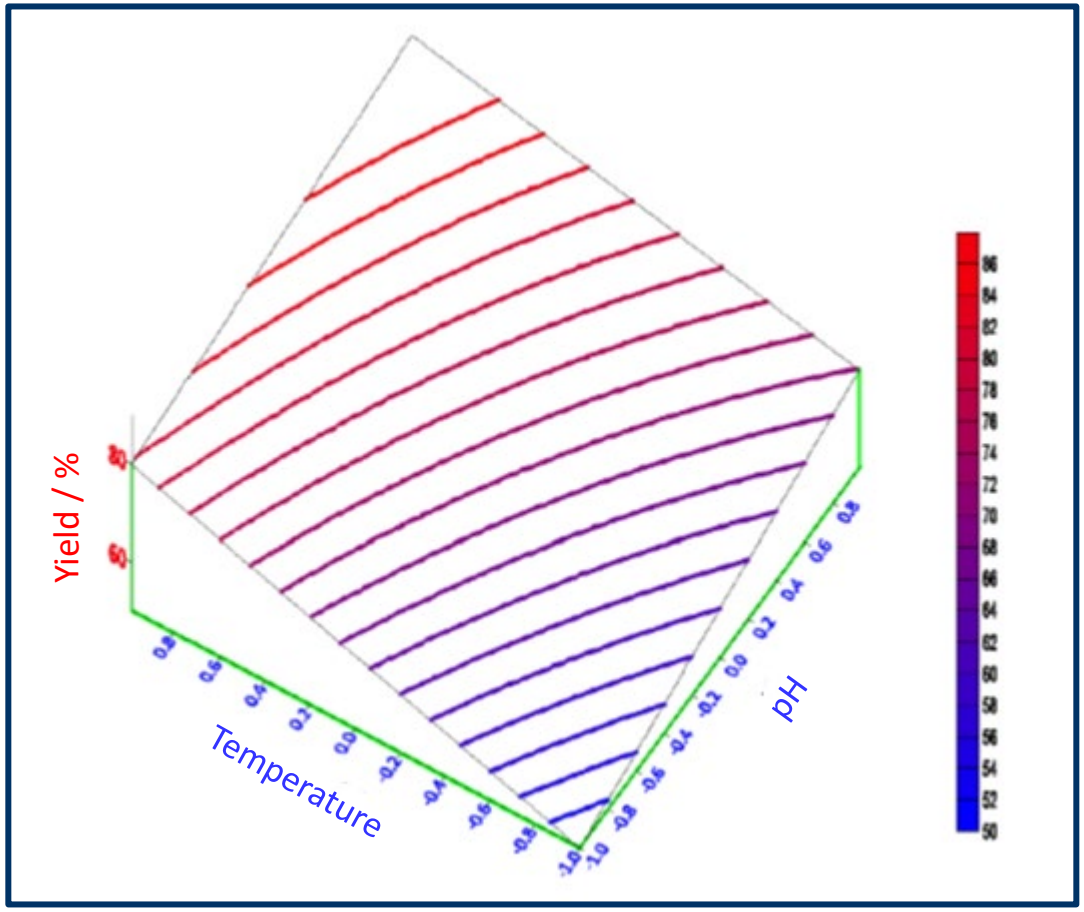
In the specific case $b_0 = (50 + 70 + 80 + 90)/4 = 72.5$

The model can thus be expressed with the equation (in which pH and T are meant to be expressed as coded variables):

$$y = 72.5 + 7.5\ pH + 12.5\ T - 2.5\ pH\ T$$

A graphical representation of the model can be obtained using a 3D plot, in which temperature and pH are still used as coded variables.

Colored lines designed on the surface correspond to iso-response curves, with response values represented on the scale drawn on the right of the 3D plot.



It is worth noting that model coefficients can be generally calculated using the following equation (in the specific example k = 2):

$$b = \frac{\sum_{i=1}^{2^k} c_i y_i}{2^k}$$

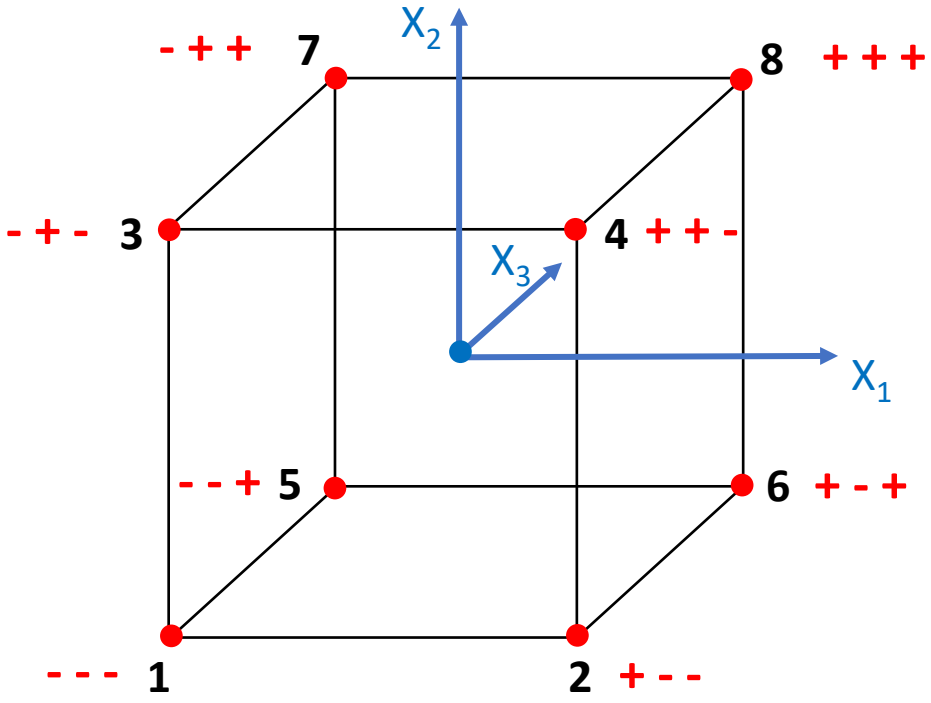
where c_i are the coefficients reported in the column of the model matrix corresponding to a specific model coefficient.

2^3 full factorial design

When a 2^3 full factorial design is considered, $2^3 = 8$ experiments have to be performed. The coded values of the three factors can be arranged in a design table, according to the Box notation, and drawn in a 3D plot:

Design table

	a	b	c
N run	x_1	x_2	x_3
1	-	-	-
2	+	-	-
3	-	+	-
4	+	+	-
5	-	-	+
6	+	-	+
7	-	+	+
8	+	+	+
	b_1	b_2	b_3



It it worth noting that in the Box notation the column referred to x_1 in the design table is represented by -1 and +1 values, alternatively. As a general rule, the column referred to the k^{th} factor is a vector with 2^{k-1} experiments at -1 level, followed by as many at +1 level.

Supposing that an interaction model is adopted:

$$y = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_{12}x_{12} + b_{13}x_{13} + b_{23}x_{23} + b_{123}x_{123}$$

a model table can be obtained from the design table, according to some rules that are included in the column algebra developed by Box:

Specifically, signs in colums related to interaction effects (ab, ac, bc and abc) are obtained by multiplications of signs present in the columns of the involved factors.

For example, in the case of run #1, the sign of the ab column will be (+), since it comes from the multiplication of two negative signs.

Model table

Design table

	a	b	c	ab	ac	bc	abc
N run	x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	x_{123}
1	-	-	-	+	+	+	-
2	+	-	-	-	-	+	+
3	-	+	-	-	+	-	+
4	+	+	-	+	-	-	-
5	-	-	+	+	-	-	+
6	+	-	+	-	+	-	-
7	-	+	+	-	-	+	-
8	+	+	+	+	+	+	+
	b_1	b_2	b_3	b_{12}	b_{13}	b_{23}	b_{123}

In order to obtain the model matrix a column containing only +1 values, related to the b_0 coefficient (as for the 2^2 factorial design), has to be introduced before the design table:

Model table

Design table

		a	b	c	ab	ac	bc	abc	
N run		x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	x_{123}	y
1	+1	-	-	-	+	+	+	-	12
2	+1	+	-	-	-	-	+	+	14
3	+1	-	+	-	-	+	-	+	68
4	+1	+	+	-	+	-	-	-	19
5	+1	-	-	+	+	-	-	+	42
6	+1	+	-	+	-	+	-	-	46
7	+1	-	+	+	-	-	+	-	71
8	+1	+	+	+	+	+	+	+	32
		b_1	b_2	b_3	b_{12}	b_{13}	b_{23}	b_{123}	

In the table responses are finally added in the last column.
In the specific case they represent reaction yields as a function of the following variables:
x_1 = temperature, x_2 = pH, x_3 = catalyst

By analogy with calculations made before for the 2^2 factorial design, the following calculations can be made:

Average level:

$b_0 = (12+14+68+19+42+46+71+32) / 8 = 38$

Main effect:

Effect of $x_1 = (14+19+46+32-12-68-42-71) / 4 = -20.5$

Effect of $x_2 = (68+19+71+32-12-14-42-46) / 4 = 19$

Effect of $x_3 = (42+46+71+32-12-14-68-19) / 4 = 19.5$

Interaction effect:

Effect of $x_1x_2 = (12+19+42+32-14-68-46-71) / 4 = -23.5$

Effect of $x_1x_3 = (12+68+46+32-14-19-42-71) / 4 = 3$

Effect of $x_2x_3 = (12+14+71+32-68-19-42-46) / 4 = -11.5$

Effect of $x_1x_2x_3 = (14+68+42+32-12-19-46-71) / 4 = 2$

Resulting model

$$y = 38 - 10.25x_1 + 9.5x_2 + 9.75x_3 - 11.75x_1x_2 + 1.5x_1x_3 - 5.75x_2x_3 + 1x_1x_2x_3$$

It can thus be inferred that interactions between x_1 and x_3 and between x_1, x_2 and x_3 are hardly significant (their coefficients are much lower than other ones).

A further example of 2^3 full factorial design

A chemical company had a problem with the viscosity of a polymer when a variation in a raw material occurred.
After performing OVAT experiments without finding a solution, a 2^3 full factorial design was implemented by selecting three variables corresponding to the amounts of three polymerization reagents (A, B and C), whose original formulation was 10, 4 and 10 g, respectively.

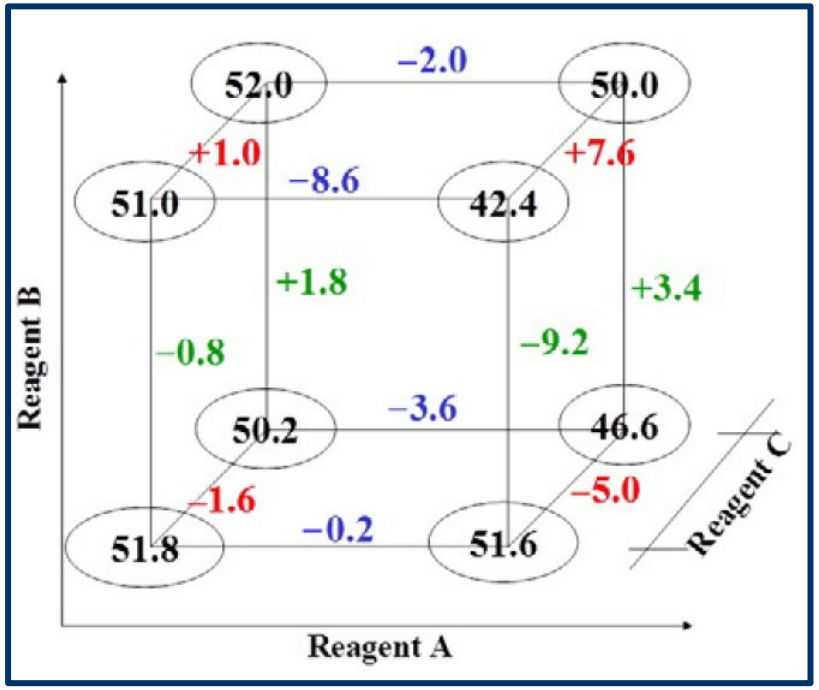
In order to keep the original recipe as the central point for each variable, levels -1 and +1 were set to correspond to 9 and 11 g for reagents A and C and 3.6 and 4.4 for reagent B.
The experimental plan was the following:

Exp.	Reagent *A* (g)	Reagent *B* (g)	Reagent *C* (g)
1	9	3.6	9
2	11	3.6	9
3	9	4.4	9
4	11	4.4	9
5	9	3.6	11
6	11	3.6	11
7	9	4.4	11
8	11	4.4	11

The model matrix was the following:

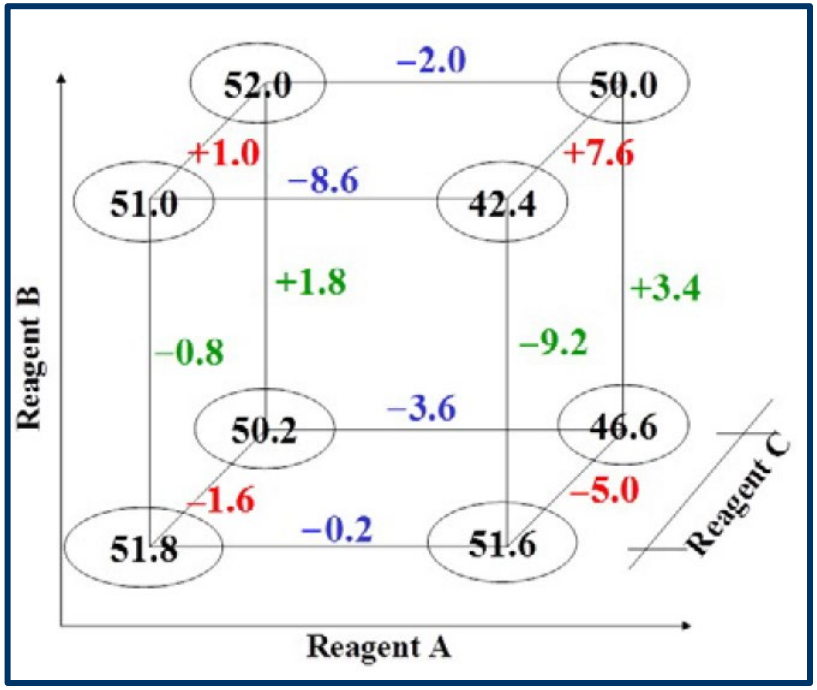
b_0	b_1	b_2	b_3	b_{12}	b_{13}	b_{23}	b_{123}	Y
1	−1	−1	−1	1	1	1	−1	51.8
1	1	−1	−1	−1	−1	1	1	51.6
1	−1	1	−1	−1	1	−1	1	51.0
1	1	1	−1	1	−1	−1	−1	42.4
1	−1	−1	1	1	−1	−1	1	50.2
1	1	−1	1	−1	1	−1	−1	46.6
1	−1	1	1	−1	−1	1	−1	52.0
1	1	1	1	1	1	1	1	50.0

A graphical representation of the experimental results is the following:



It can be clearly seen that all the experiments performed at a lower value of reagent A led to responses greater than the threshold value (corresponding to 46).
It can therefore be said that by lowering the amount of A an increase of the response is obtained.

As for reagent B, it can be seen that its increase leads to a decrease of the response when reagent C is at a lower level and to an increase of the response when reagent C is at a higher level.
This is a clear example of interaction between two variables.



The same interaction is detected when taking into account reagent C. It can be seen that an increase of reagent C improves the response whereas a decrease occurs when reagent B is at a lower level.

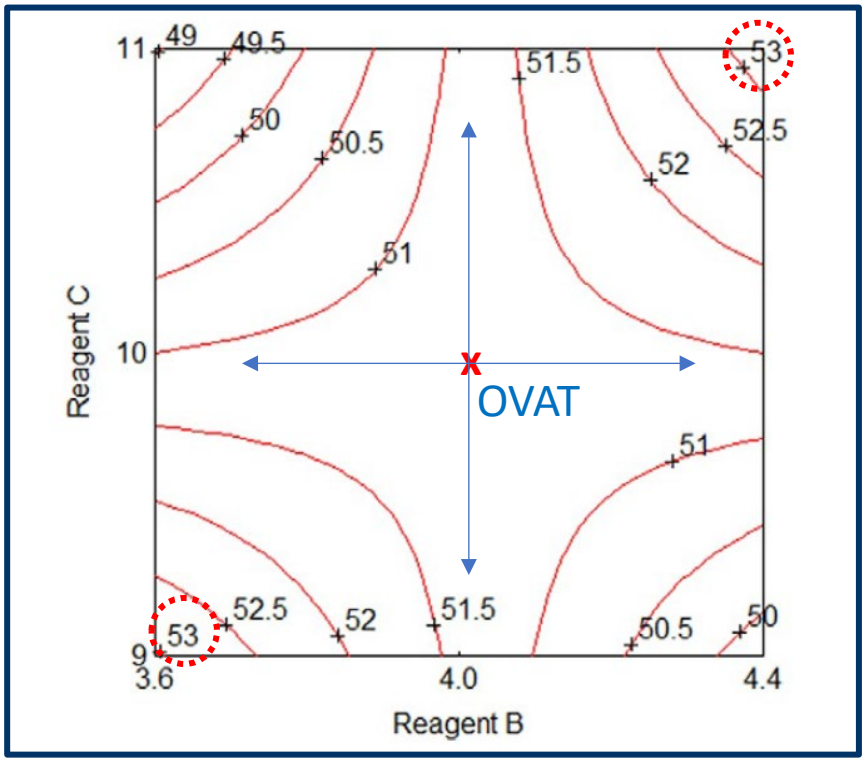
The following model is obtained:

$$Y = 49.4 - 1.8X_1 - 0.6X_2 + 0.2X_3 - 0.8X_1X_2 + 0.4X_1X_3 + 1.9X_2X_3 + 1.2X_1X_2X_3$$

As apparent, coefficients related to reagent A (X_1) and to the interaction between reagents B and C (X_2, X_3) are larger than the other ones.
The negative value of the X_1 coefficient accounts for the decrease of polymer viscosity when the amount of reagent A is increased.

The interaction between reagents B and C can be interpreted by looking at the iso-response plot shown in the figure, referred to an amount of reagent A of 9 g.

It can be seen that an increase of reagent B leads to decrease viscosity when reagent C is at its lower level, while it has the opposite effect when reagent C is at its higher level.
In the same way, an increase of reagent C decreases viscosity when reagent B is at its lower level, while it has the opposite effect when reagent B is at its higher level.



Looking at the plot, it can also be understood why the OVAT approach did not produce any good result. In fact, starting from the central point (corresponding to the original formulation) and changing the amount of either reagent B or reagent C (but not both at the same time) nothing changes.
Instead, due to the strong interaction, relevant variations are observed only when both variables are changed at the same time.

As apparent from the plot, two combinations produce the same response (namely, a viscosity of 53): 3.6 g of reagent B and 9 g of reagent C and 4.4 g of reagent B and 11 g of reagent C.

As a higher amount of reagents increases the speed of the reaction, and therefore the final throughput, the latter has been selected as the best combination.

When transferred from the laboratory scale to the industrial plant the combination still proved to provide a viscosity value well over the acceptability value.

It could have been worthwhile to check if the effect was the same also outside the experimental domain, thus evaluating the possibility of obtaining even better results, yet the immediate goal was the return to the production of an acceptable product.

The main problem with the previous design was that, as there were no degrees of freedom and no previous estimate of the experimental error was available, it was not possible to determine which coefficients (factors) were statistically significant.

A different type of factorial design, including also replicates, should have been considered to perform this evaluation.

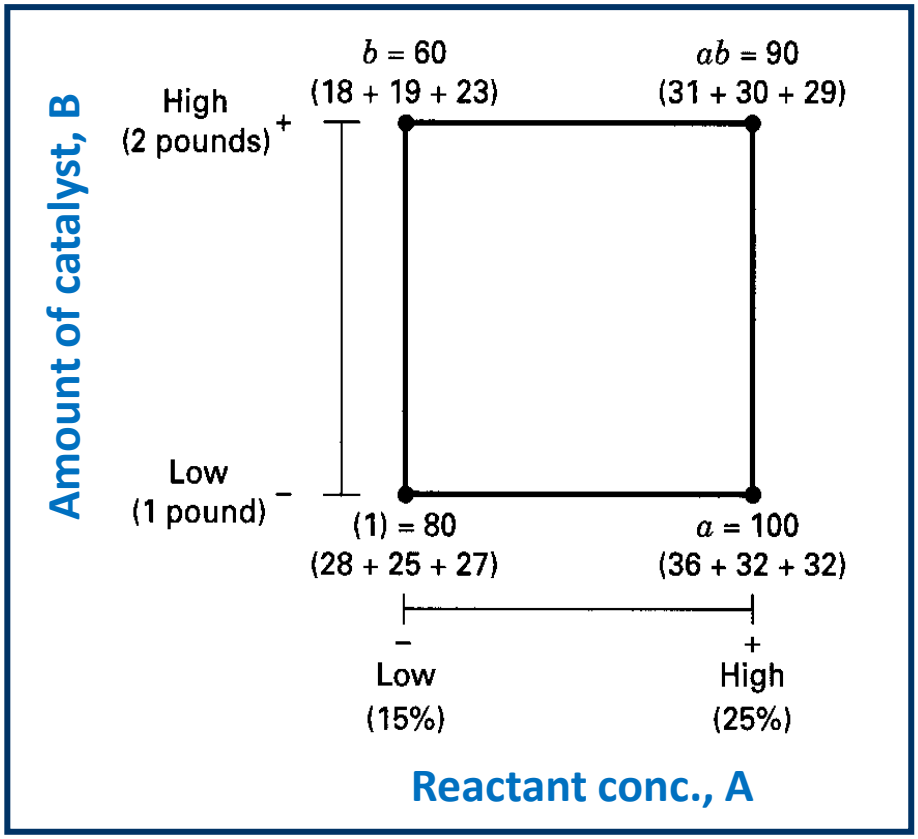
Full factorial designs with replicates: 2^2 design with 3 replicates

As an example of full factorial design with replicates let us consider a design with two two-level factors (2^2) and 3 replicates (n = 3) for each combination.

Factor A: reagent concentration (%) – levels: 15 (low) and 25 (high)
Factor B: catalyst amount (pounds) – levels: 1 (low) and 2 (high) pounds

A total of 4 × 3 = 12 experiments is performed and the following data are obtained:

The three values reported near each vertex correspond to replicated measurements of reaction yield.



Data can be conveniently summarized using the following table:

Model matrix

Model table

Design table

Interaction model

		factor			Y (replicated response)			
Exp.	**I**	**A(x_1)**	**B(x_2)**	**AB**	**1**	**2**	**3**	Row total (T_i)
1	+1	-1	-1	+1	28	25	27	80
2	+1	+1	-1	-1	36	32	32	100
3	+1	-1	+1	-1	18	19	23	60
4	+1	+1	+1	+1	31	30	29	90
								T=330

c_i

Each response can thus be classified according to two indexes, i and j:

- ✓ i indicates the number of experiment, in turn corresponding to a specific combination of levels, thus i = 1, 2, …, 2^k;
- ✓ j indicates the number related to the replicate, thus j = 1, 2, …, n.

The average effect of factor A (x_1) is calculated as follows:

(-80+100-60+90)/[2(3)] = 50/[2(3)] = 8.33

In general terms the average effect can be calculated with the following formulas:

$$\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{n} c_i y_{i,j}}{n2^{k-1}} = \frac{\sum_{i=1}^{2^k} c_i T_i}{n2^{k-1}} = \frac{\sum_{i=1}^{2^k} c_i \bar{y}_i}{2^{k-1}}$$

where T_i correspond to row totals.

Notably, the term at the numerator in the second expression corresponds to the contrast of response values:

$$Contrast = \sum_{i=1}^{2^k}\sum_{j=1}^{n} c_i y_{i,j}$$

As a general definition, contrast is any linear combination of values for which the sum of the coefficients is zero.

By analogy with the previous calculation, the average effect of factor B (x_2) is calculated as follows:

(-80-100+60+90)/[2(3)] = -30/[2(3)] = -5.00

Finally, the average effect of the interaction AB (x_1x_2) is calculated as follows:

(80-100-60+90)/[2(3)] = 10/[2(3)] = 1.67

Since replicates were performed in this case, the significance of effects can be evaluated using ANOVA.

Sum of squares can be calculated according to the following equation: $$SS = \frac{(\sum_{i=1}^{2^k} \sum_{j=1}^{n} c_i y_{i,j})^2}{n \sum_{i=1}^{2^k} c_i^2}$$

thus specific sum of squares are the following:

$$SS_A = \frac{(50)^2}{4(3)} = 208.33 \qquad SS_B = \frac{(-30)^2}{4(3)} = 75.00 \qquad SS_{AB} = \frac{(10)^2}{4(3)} = 8.33$$

The total sum of squares can be calculated according to one of the equations described in the ANOVA slides:

$$SS_{tot} = \sum_i \sum_j y_{i,j}^2 - \frac{T^2}{N}$$ thus:

SS_{tot} = 9398 - (330)²)/[4(3)] = 323.00

The error sum of squares SS_E can be calculated as a difference:

$$\begin{aligned} SS_E &= SS_T - SS_A - SS_B - SS_{AB} \\ &= 323.00 - 208.33 - 75.00 - 8.33 \\ &= 31.34 \end{aligned}$$

The ANOVA table can thus be written as follows:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
A	208.33	1	208.33	53.15	0.0001
B	75.00	1	75.00	19.13	0.0024
AB	8.33	1	8.33	2.13	0.1826
Error	31.34	8	3.92		
Total	323.00	11			

Notably, the degrees of freedom related to the error are calculated as the difference between the total degrees of freedom and the sum of those related to the other sources of variation. Anyway, the general equation is $2^k(n-1)$.

Based on the F_0 values, obtained by ratioing MS_A, MS_B and MS_{AB} by MS_E, the effects due to factors A and B are significant, whereas the effect of the AB interaction is not significant (indeed, the P-value is greater than 0.05).

Regression model

The results of the experiment described in previous slides can be expressed in terms of the following model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$$

Notably, the term including the interaction is absent, since this was not considered significant after ANOVA.

The model is fitted by the following equation:

$$\hat{y} = b_0 + b_1 x_1 + b_2 x_2$$

Regression coefficients b_1 and b_2 are calculated as the average effects observed for the respective factors divided by 2 (since coded variables with a total range of +1 – (-1) = 2 were adopted):

$$b = \frac{\sum_{i=1}^{2^k} c_i T_i}{n 2^k}$$

As described before, the b_0 coefficient is the grand average of all the 12 observations.

The final equation is thus:

$$\hat{y} = 27.5 + \left(\frac{8.33}{2}\right) x_1 + \left(\frac{-5.00}{2}\right) x_2$$

It can be used to obtain the predicted (fitted) values of y at the four points in the design and then calculate the residuals, i.e. the differences between the observed and the fitted values of y.

For example, when both the reactant concentration and the catalyst are at the low level ($x_1 = x_2 = -1$) the predicted yield is:

$\hat{y}$ = 27.5 - (8.33/2) - (-5.00/2) = 25.835

Since three responses were obtained at those levels, i.e., 28, 25 and 27, the residuals are:

$$e_1 = 28 - 25.835 = 2.165$$
$$e_2 = 25 - 25.835 = -0.835$$
$$e_3 = 27 - 25.835 = 1.165$$

The other predicted values and the corresponding residuals are:

high level of the reactant concentration and low level of the catalyst

$$\hat{y} = 27.5 + \left(\frac{8.33}{2}\right)(+1) + \left(\frac{-5.00}{2}\right)(-1) = 34.165$$

$$e_4 = 36 - 34.165 = 1.835$$
$$e_5 = 32 - 34.165 = -2.165$$
$$e_6 = 32 - 34.165 = -2.165$$

low level of the reactant concentration and high level of the catalyst

$$\hat{y} = 27.5 + \left(\frac{8.33}{2}\right)(-1) + \left(\frac{-5.00}{2}\right)(+1) = 20.835$$

$$e_7 = 18 - 20.835 = -2.835$$
$$e_8 = 19 - 20.835 = -1.835$$
$$e_9 = 23 - 20.835 = 2.165$$

high level of the reactant concentration and high level of the catalyst

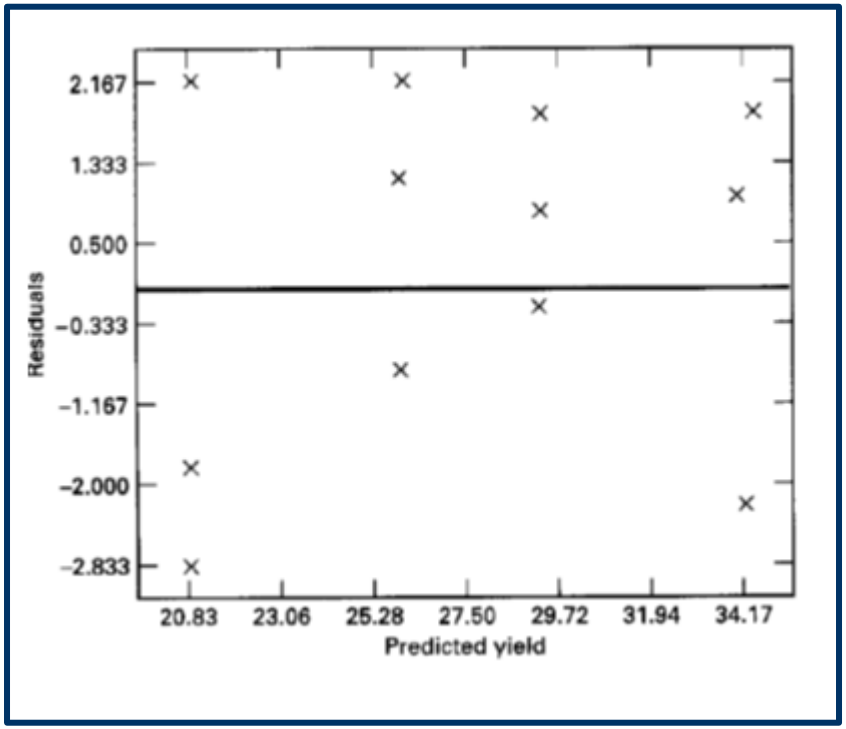
$$\hat{y} = 27.5 + \left(\frac{8.33}{2}\right)(+1) + \left(\frac{-5.00}{2}\right)(+1) = 29.165$$

$$e_{10} = 31 - 29.165 = 1.835$$
$$e_{11} = 30 - 29.165 = 0.835$$
$$e_{12} = 29 - 29.165 = -0.165$$

Residuals can be plotted versus predicted yield:

as apparent, the residuals are distributed as negative or positive values without any prevalence of either the former or the latter in a specific interval of the predicted yield.

Therefore, there is no reason to suspect problems with the validity of the model conclusions.



The regression model:

$$\hat{y} = 27.5 + \left(\frac{8.33}{2}\right) x_1 + \left(\frac{-5.00}{2}\right) x_2$$

can be also used to generate response surface plots.

It is usually desirable to construct these plots in terms of natural factor levels, which can be obtained simply by replacing coded variables x_1 and x_2 with the relationships between them and the corresponding natural factors, i.e. the reactant and the catalyst concentrations, respectively (note that coded values = [Uncoded-Average]/Half-width of range):

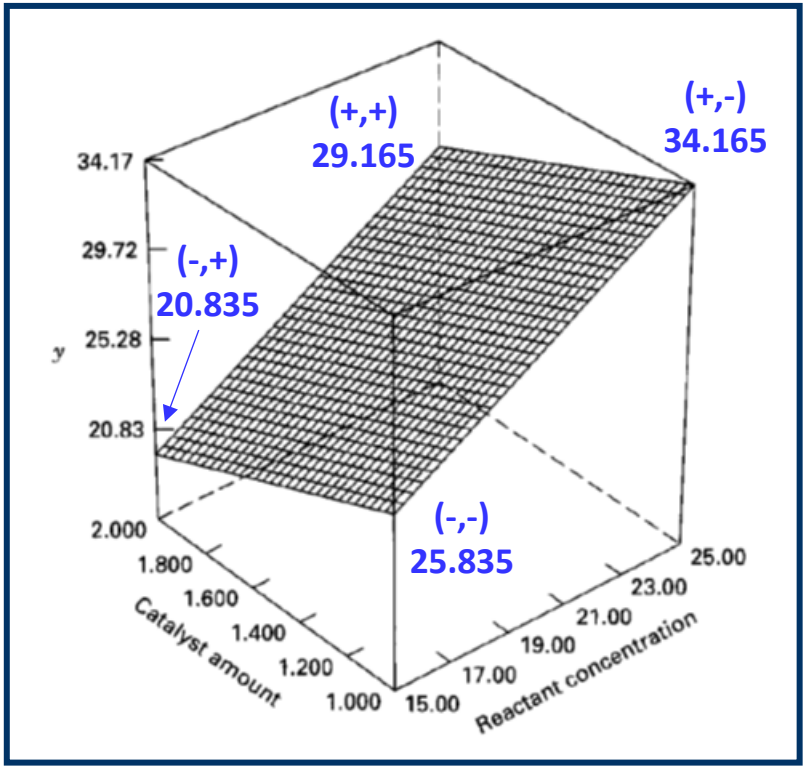
$$\hat{y} = 27.5 + \left(\frac{8.33}{2}\right)\left(\frac{\text{Conc} - 20}{5}\right) + \left(\frac{-5.00}{2}\right)\left(\frac{\text{Catalyst} - 1.5}{0.5}\right)$$

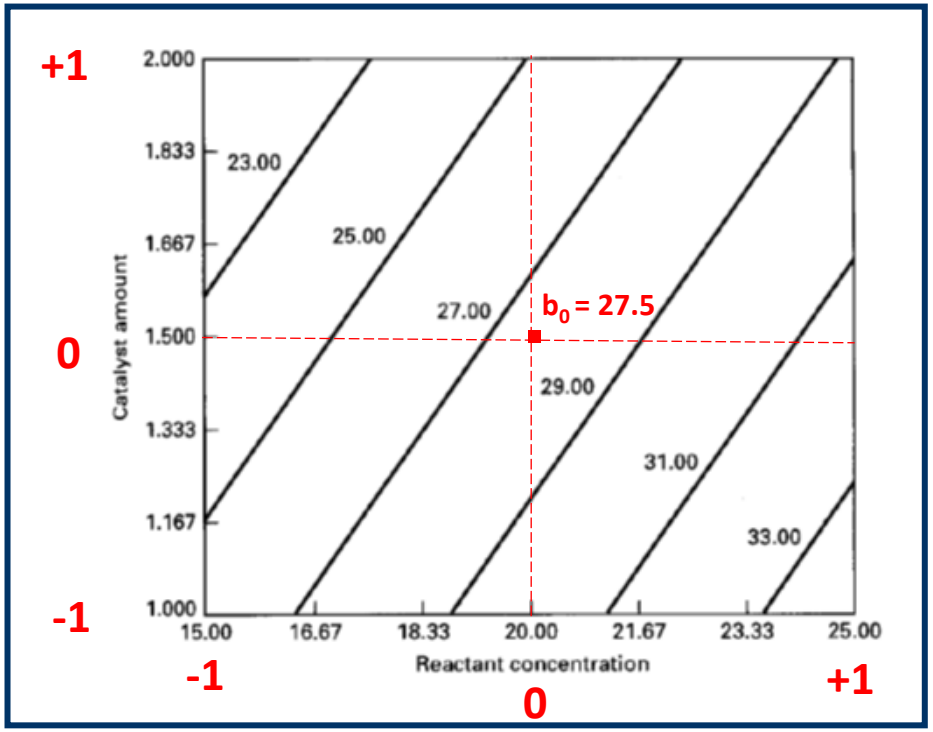
$$\hat{y} = 18.34 + 0.833\ \text{Conc} - 5.00\ \text{Catalyst}$$

The response surface related to the model can be drawn in a 3D plot:

Since the model is a first-degree one, the fitted response surface is a plane.

The plot clearly shows that the reaction yield increases as the reactant concentration increases and as the catalyst amount decreases.





Another possible representation of the same information is the bidimensional contour plot.

Full factorial designs with replicates: 2^3 design with 2 replicates

As a further example of full factorial design with replicates let us consider a design with three two-level factors (2^3) and 2 replicates (n = 2) for each combination, related to the optimization of a HPLC mobile phase composition.

Factor A: pH of mobile phase
Factor B: counterion concentration
Factor C: % of organic modifier

In this case the response is represented by the capacity factor $k' = (t_r - t_m) / t_m$.

A total of 8 × 2 = 16 experiments is performed, in random order, and the following data table (where only average responses are shown, for simplicity) is obtained:

Exp.	x_1	x_2	x_3	Average response
1	-1	-1	-1	4.7
2	+1	-1	-1	9.9
3	-1	+1	-1	7.0
4	+1	+1	-1	15
5	-1	-1	+1	2.7
6	+1	-1	+1	5.3
7	-1	+1	+1	3.2
8	+1	+1	+1	6.0

The following model is hypothesised:

$$y = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_{12}x_1x_2 + b_{13}x_1x_3 + b_{23}x_2x_3 + b_{123}x_1x_2x_3$$

The table summarizing the model matrix and average responses for the 8 experiments is the following:

Exp	Model matrix								$\overline{y}_i$
	mean	X_1	X_2	X_3	X_1X_2	X_1X_3	X_2X_3	$X_1X_2X_3$	
1	+1	-1	-1	-1	+1	+1	+1	-1	4.7
2	+1	+1	-1	-1	-1	-1	+1	+1	9.9
3	+1	-1	+1	-1	-1	+1	-1	+1	7.0
4	+1	+1	+1	-1	+1	-1	-1	-1	15
5	+1	-1	-1	+1	+1	-1	-1	+1	2.7
6	+1	+1	-1	+1	-1	+1	-1	-1	5.3
7	+1	-1	+1	+1	-1	-1	+1	-1	3.2
8	+1	+1	+1	+1	+1	+1	+1	+1	6.0
	b_0	b_1	b_2	b_3	b_{12}	b_{13}	b_{23}	b_{123}	

Average effects of factors are calculated as follows:

Factor x_1

(-4.7+9.9-7.0+15-2.7+5.3-3.2+6.0)/4 = 18.6/4 = 4.65

Factor x_2

(-4.7-9.9+7.0+15-2.7-5.3+3.2+6.0)/4 = 8.6/4 = 2.15

Factor x_3

(-4.7-9.9-7.0-15+2.7+5.3+3.2+6.0)/4 = -19.4/4 = - 4.85

Interactions

X_1X_2: 0.75

X_1X_3: -1.95

X_2X_3: -1.55

$X_1X_2X_3$: -0.65

The significance of effects can be evaluated using ANOVA, starting from the calculation of the sums of squares:

$$\mathbf{SS} = \frac{(\sum_{i=1}^{2^k}\sum_{j=1}^{n} c_i y_{i,j})^2}{n\sum_{i=1}^{2^k} c_i^2} = \frac{(contrast)^2}{n\sum_{i=1}^{2^k} c_i^2} = \frac{(contrast)^2}{n2^k}$$

Since the average effect of a factor is expressed as: $\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{n} c_i y_{i,j}}{n2^{k-1}}$

an equation correlating a SS value with the average effect of the corresponding factor can be obtained:

$$\mathbf{SS} = \frac{(\sum_{i=1}^{2^k}\sum_{j=1}^{n} c_i y_{i,j})^2}{n\sum_{i=1}^{2^k} c_i^2} = \frac{n\left(2^{k-1}\right)^2[\text{average effect}]^2}{2^k}$$

In the present case each SS is thus obtained by multiplying the square of the corresponding average effect by $2 * (2^2)^2 / 8 = 4$.

The following ANOVA table is thus obtained:

Source	SS	df	MS	F_0
X_1	86.49	1	86.49	4324.5
X_2	18.49	1	18.49	924.5
X_3	94.09	1	94.09	4704.5
X_1X_2	2.25	1	2.25	112.5
X_1X_3	15.21	1	15.21	760.5
X_2X_3	9.61	1	9.61	480.5
$X_1X_2X_3$	1.69	1	1.69	84.5
Error	0.160	8	0.02	

where, as explained previously, SS_E and the respective degrees of freedom (df) are obtained by difference, considering that :

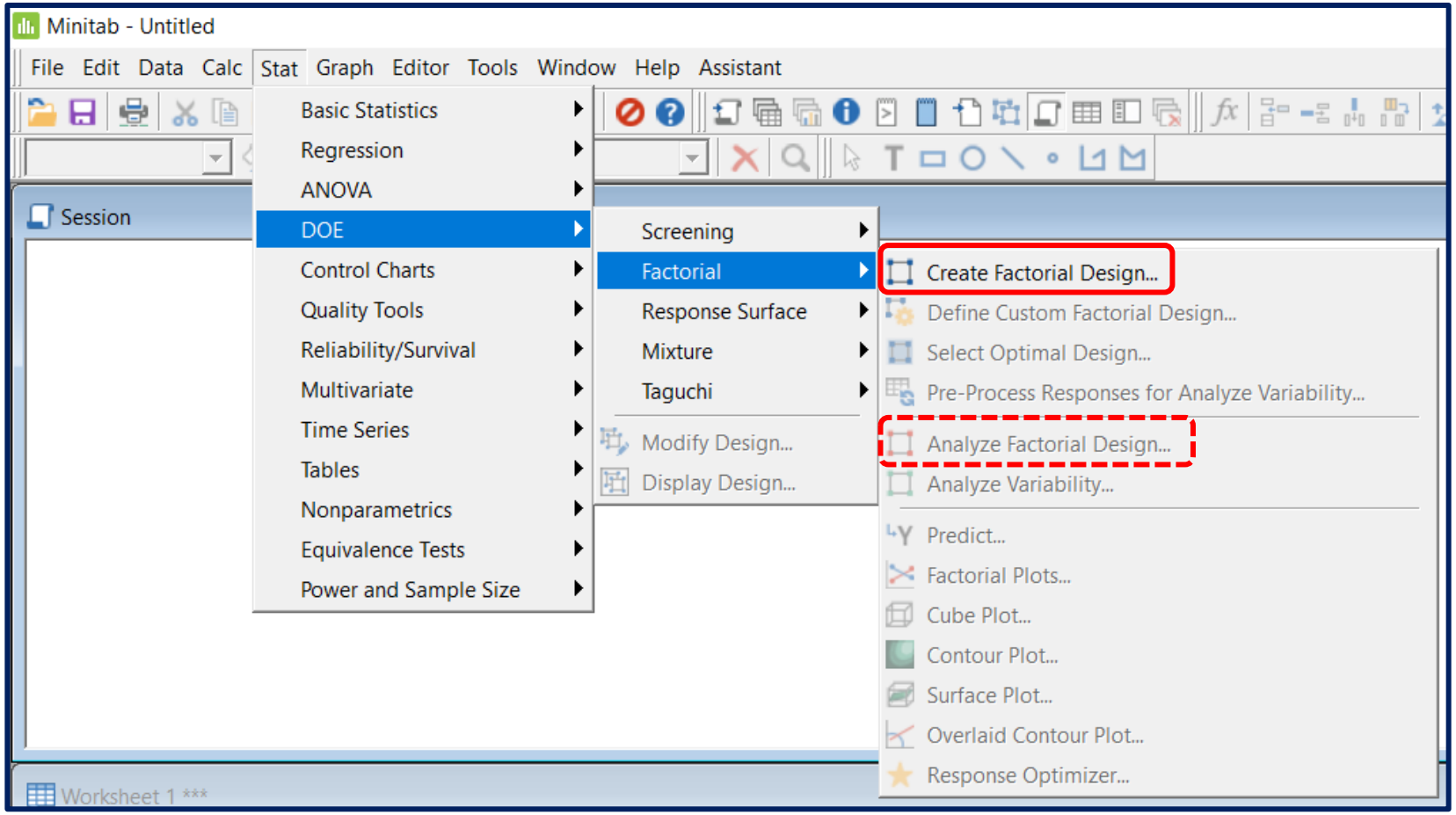
$$SS_{tot} = \sum_i \sum_j y_{i,j}^2 - \frac{T^2}{N} = 227.990$$

Since all F_0 values are higher than the critical value $F_{1,8}$ (7.57) all factors have a significant effect.

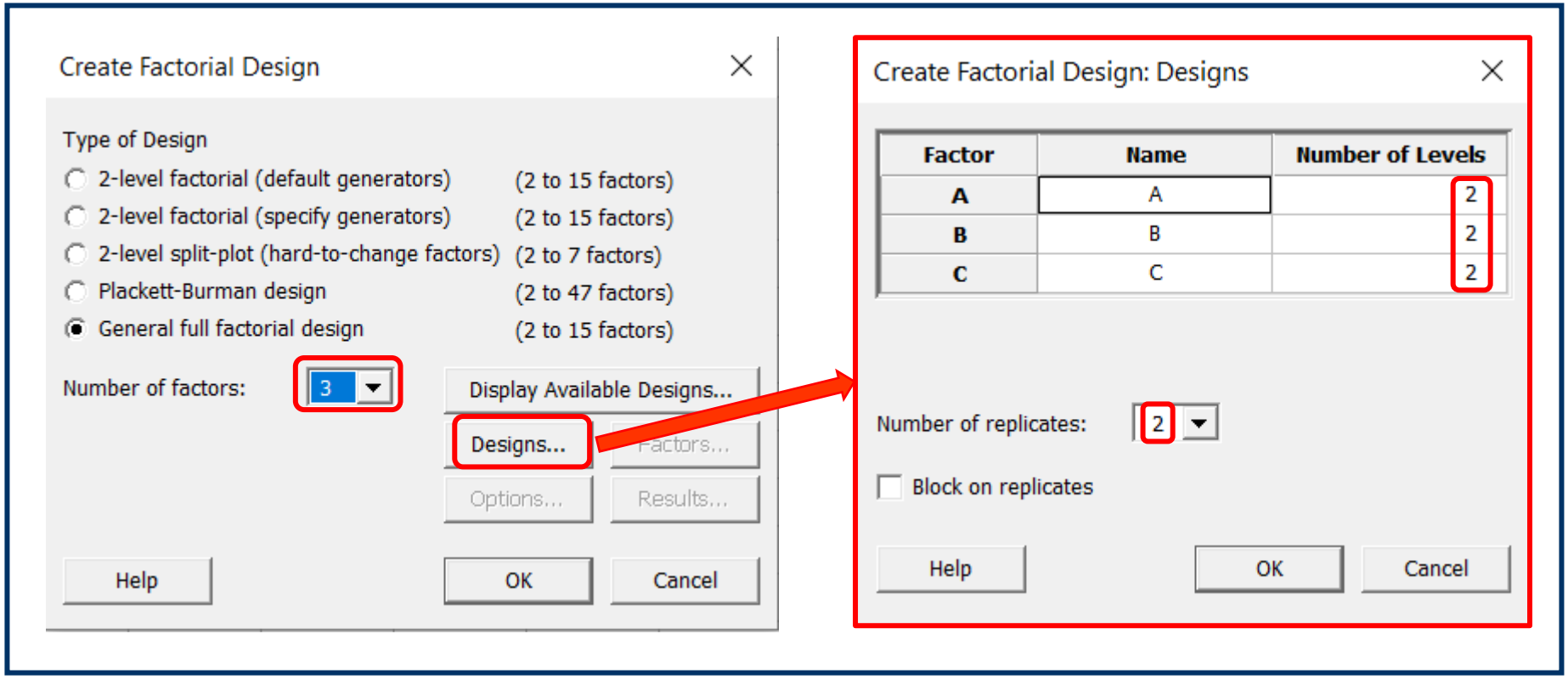
Use of Minitab 18 for calculations of a full 2^3 factorial design with replicates

The Minitab 18 software can be used to develop a full factorial design with replicates.

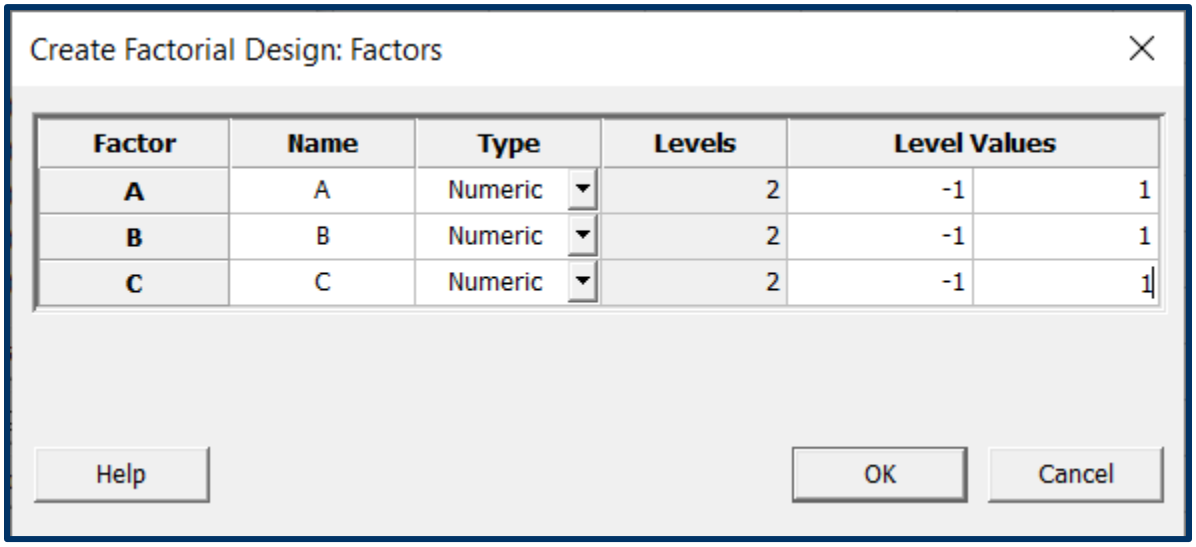
The option is accessed from the Stat > DOE > Factorial > Create Factorial Design… path.



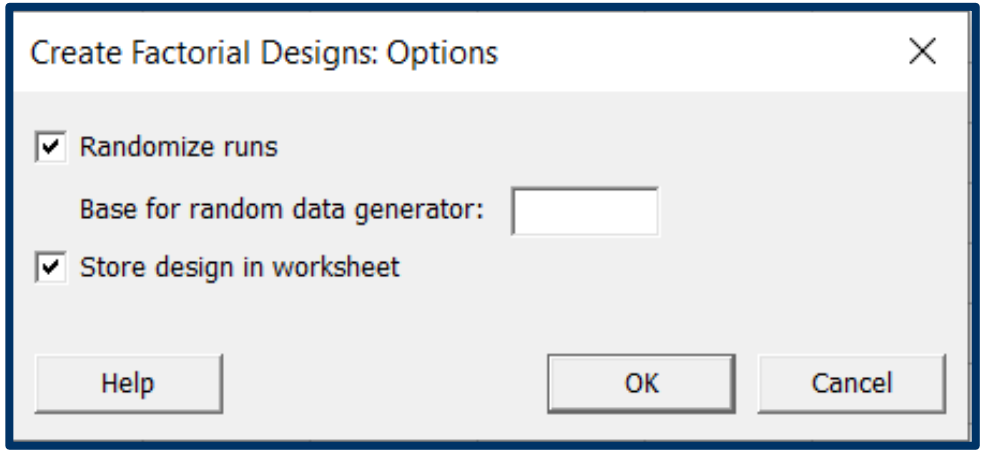
Several different designs can be set, including a general full factorial design. The number of factors (3 in the specific case) is indicated, then details on the numbers of levels and of replicates are reported in the Designs… window.



The Factors… window enables the specification of factors names and level values.
In the specific example coded values are assigned to the levels:



The Options… window enables the selection of runs randomization and the storage of design in worksheet:



When the OK command is given in the Create factorial design window, this option leads to the automatic insertion of several data into the first seven columns of the worksheet.

The first column indicates the randomized order of runs, whereas columns 5, 6 and 7 indicate the coded values of factors to be used in the respective run:

Worksheet 1 ***

	C1	C2	C3	C4	C5	C6	C7	C8
	StdOrder	RunOrder	PtType	Blocks	A	B	C	Response
1	2	1	1	1	-1	-1	1	2.6
2	12	2	1	1	-1	1	1	3.1
3	6	3	1	1	1	-1	1	5.2
4	9	4	1	1	-1	-1	-1	4.6
5	14	5	1	1	1	-1	1	5.4
6	15	6	1	1	1	1	-1	14.9
7	7	7	1	1	1	1	-1	15.1
8	16	8	1	1	1	1	1	5.9
9	11	9	1	1	-1	1	-1	6.9
10	8	10	1	1	1	1	1	6.1
11	3	11	1	1	-1	1	-1	7.1
12	13	12	1	1	1	-1	-1	9.8
13	1	13	1	1	-1	-1	-1	4.8
14	5	14	1	1	1	-1	-1	10.0
15	4	15	1	1	-1	1	1	3.3
16	10	16	1	1	-1	-1	1	2.8

Notably, column 3 indicates the type of points (1 is the code for corner points, *i.e.*, those referred to extreme values of the coded variables, -1 or 1); column 4 is referred to eventual blocks, i.e., groups of runs between which an eventual difference can be observed (e.g., groups of replicated runs obtained in different days). In the present case all runs were included in a single block.

As soon as a run is completed the Response can be introduced in column 8.

Once all data are introduced, the design analysis can be made through the pathway:

Stat > DoE > Factorial Design > Analyze Factorial Design…

Different types of information can be obtained after the OK command is given.

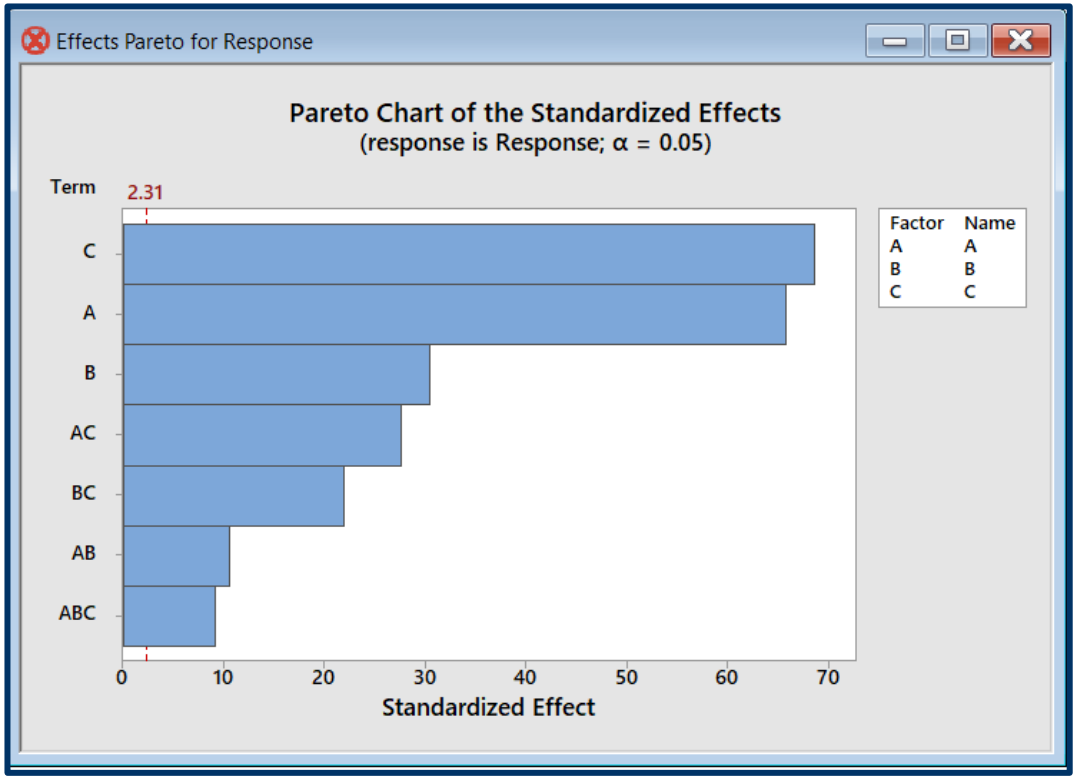
As an example, the ANOVA table can be easily obtained:

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Model	7	227.830	32.5471	1627.36	0.000
Linear	3	199.070	66.3567	3317.83	0.000
A	1	86.490	86.4900	4324.50	0.000
B	1	18.490	18.4900	924.50	0.000
C	1	94.090	94.0900	4704.50	0.000
2-Way Interactions	3	27.070	9.0233	451.17	0.000
A*B	1	2.250	2.2500	112.50	0.000
A*C	1	15.210	15.2100	760.50	0.000
B*C	1	9.610	9.6100	480.50	0.000
3-Way Interactions	1	1.690	1.6900	84.50	0.000
A*B*C	1	1.690	1.6900	84.50	0.000
Error	8	0.160	0.0200		
Total	15	227.990			

The so-called Pareto Chart of the Standardized Effects is also provided. This plot shows the absolute values of the standardized effects, ordered from the largest to the smallest one. Each value is obtained by multiplying the average effect of a specific factor by $(MSE)^{1/2}$ and then by 100.

The chart also plots a reference line to indicate which effects are statistically significant, i.e., those whose bars cross the reference line (all the effects in the specific case, as indicated also by the ANOVA table). This outcome is based on a t-test; in the present case, the reference line is located at 2.31, corresponding to the Student's t value for $1-\alpha = 0.975$ and $n = 8$.



Fractional factorial designs

Fractional factorial designs (FFD) are experimental designs consisting of a carefully chosen subset (fraction) of the experimental runs of a full factorial design.

They are applied when the number of experiments required for a full factorial design is too high and exploit the fact that many experiments in full factorial designs are often redundant, giving little or no new information about the system.

When two-levels factors are considered, FFDs are indicated as 2^{k-p} designs, where p represents the size of the fraction adopted.

Let us consider a 2^3 full factorial design, which would imply the following table:

Model table

Design table

Exp.	I	1	2	3	12	13	23	123	Y
1	+1	-1	-1	-1	+1	+1	+1	-1	38
2	+1	+1	-1	-1	-1	-1	+1	+1	37
3	+1	-1	+1	-1	-1	+1	-1	+1	26
4	+1	+1	+1	-1	+1	-1	-1	-1	24
5	+1	-1	-1	+1	+1	-1	-1	+1	30
6	+1	+1	-1	+1	-1	+1	-1	-1	28
7	+1	-1	+1	+1	-1	-1	+1	-1	19
8	+1	+1	+1	+1	+1	+1	+1	+1	16

Model matrix

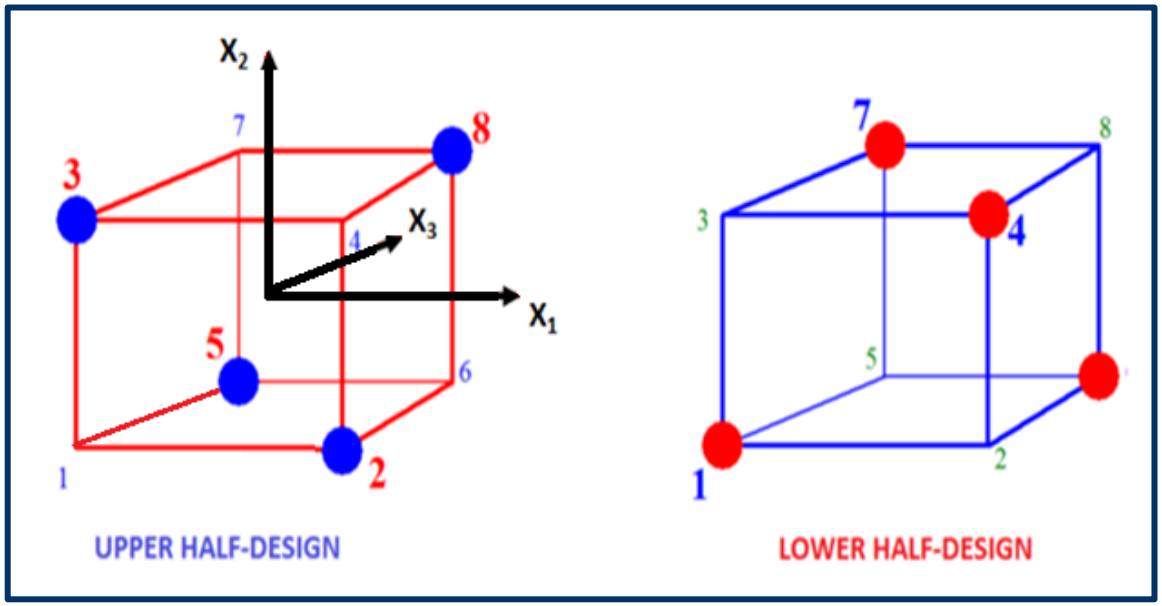
The following effects can be calculated, based on responses obtained from the different runs:

As apparent, interactions have a very low effect.

I	27.25
1	-1
2	-6
3	-4
12	-0.25
13	-0.25
23	0.25
123	0.00

A possible fractional design could thus be obtained by setting p = 1, i.e., by performing only $2^{3-1} = 4$ runs, instead of 8.

Two of the possibile sub-sets of 4 runs arising from the 2^3 design are shown in the figure on the right:



It is easy to note that, if a new design table is drawn, based on runs 5, 2, 3 and 8, i.e., on the upper-half-design, signs observed for factor 3 are equal to those that would be obtained for a 12 interaction in a 2^2 full factorial design:

Box notation

Exp.	I	1	2	3	Y
5	+1	-1	-1	+1	30
2	+1	+1	-1	-1	37
3	+1	-1	+1	-1	26
8	+1	+1	+1	+1	16
		x_1	x_2	x_{12}	

2^2 model matrix

Interestingly, the effects calculated with the 2^3 full factorial design and the 2^2 fractional design are comparable, within a reasonable uncertainty:

Effects	2^3	2^2
I	27.25	27.25
1	-1	-0.75
2	-6	-6.25
3	-4	-4.25

More specifically, if the effects of factor 3 and of the 12 interaction for the 2^3 full factorial design are summed the result is identical to the effect due to factor 3 in the 2^2 fractional design. By analogy, the sums of effects of factors I and 123, 1 and 23, 2 and 13 of the full factorial design corresponds, respectively, to the effects of factors I, 1 and 2 of the fractional design.

In other words, effects calculated with a fractional design are mixed (confused); for this reason, they are also called «aliased» and are indicated with λ_i, to be distinguished from pure effects, indicated as E_i.

However, if the effects due to interactions are small, like in this case, the result of using a fractional design instead of a full one can be acceptable.

A «confusion table», also called «table of alias», can be drawn to emphasize the correspondence between effects related to a full and to a fractional factorial design:

PURE EFFECTS	FULL DESIGN	ALIAS EFFECTS	FRACTIONAL DESIGN	ALIASED
E_I	27.25	λ_I	27.25	E_I E_{123}
E_1	-1	λ_1	-0.75	E_1 E_{23}
E_2	-6	λ_2	-6.25	E_2 E_{13}
E_3	-4	λ_3	-4.25	E_3 E_{12}
E_{12}	-0.25			
E_{13}	-0.25			
E_{23}	0.25			
E_{123}	0			

The correspondence existing between pure effects in the full design and aliased effects in fractional ones, like the upper/lower half designs, can be appreciated using a color code:

	Exp.	I	1	2	3	12	13	23	123	Y
Upper half design	5	+1	-1	-1	+1	+1	-1	-1	+1	30
	2	+1	+1	-1	-1	-1	-1	+1	+1	37
	3	+1	-1	+1	-1	-1	+1	-1	+1	26
	8	+1	+1	+1	+1	+1	+1	+1	+1	16
Lower half design	1	+1	-1	-1	-1	+1	+1	+1	-1	38
	6	+1	+1	-1	+1	-1	+1	-1	-1	28
	7	+1	-1	+1	+1	-1	-1	+1	-1	19
	4	+1	+1	+1	-1	+1	-1	-1	-1	24

In particular, alias have the same sign in the upper half design and an opposite sign in the lower half design. The following equations can be written:

$$\lambda_I = E_I + E_{123} \qquad \lambda_2 = E_2 + E_{13}$$

$$\lambda_1 = E_1 + E_{23} \qquad \lambda_3 = E_3 + E_{12}$$

Some considerations can be made also about the response model.
Indeed, for a 2^3 full factorial design the model would be the following:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3 + \varepsilon$$

The following relations are defined when the fractional design is considered:

$$x_1 = x_2 x_3$$
$$x_2 = x_1 x_3$$
$$x_3 = x_1 x_2$$
$$I = x_1 x_2 x_3$$

After their introduction in the complete model, the following new model is obtained:

$$y = \beta_0 + \beta_{123} + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_3 + \beta_{13} x_2 + \beta_{23} x_1 + \varepsilon$$

$$y = (\beta_0 + \beta_{123}) + (\beta_1 + \beta_{23}) x_1 + (\beta_2 + \beta_{13}) x_2 + (\beta_3 + \beta_{12}) x_3 + \varepsilon$$

Four coefficients can then be obtained as estimates of model parameters:

$$y = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$

$$b_0 = \beta_0 + \beta_{123}$$
$$b_1 = \beta_1 + \beta_{23}$$
$$b_2 = \beta_2 + \beta_{13}$$
$$b_3 = \beta_3 + \beta_{12}$$

Due to the adoption of a fractional design, coefficients estimate a specific true parameter but with a "contamination" due to one of the interactions.

As an example, b_1 is an estimate of β_1 but with a contamination due to the parameter related to the 23 interaction, β_{23}.

b_1 is thus named an "alias" of the two confounded coefficients.

Generators of fractional factorial designs

A general approach can be defined to determine the confounding effects. It is based on the definition of the so-called alias generators.

If the example of the upper-half design for the 2^{3-1} fractional design is considered, the fact that signs related to factor 3 are identical to those related to the 12 interaction can be expressed, in terms of Box algebra, as **3 = 12**.

Now, if both sides of this equation are multiplied by 3, the following equation is obtained:

3 • 3 = 12 • 3

Since **3 • 3 = I**, according to one of the Box algebra rules, the equation becomes: I = 123.

This relation is called the "design generator". Indeed, if each column in the matrix model is multiplied by the generator, the relations indicating confounding effects are obtained:

1.I = 1.123 = I.23 ⇒ **1 = 23**

2.I = 2.123 = 2.213 = I.13 ⇒ **2 = 13**

3.I = 3.123 = 3.312 = I.12 ⇒ **3 = 12**

Resolution for fractional factorial designs

Resolution is an important property of a fractional factorial design, since it represents its ability to separate main effects and low-order interactions from one another.

By definition, Resolution I is that of a 2^{1-1} design, *i.e.*, a design based on one factor and only on one run, which, of course, cannot even distinguish between the high and low levels of the factor under study.

Resolution II is that of a 2^{2-1} design in which the design generator is I = 12. In this case main effects are confounded with other main effects (in fact: **1.12 = 2 and 2.12 = 1)**

Resolution III is referred to a 2^{3-1} design with a design generator I = 123, like in the example discussed before. In this case main effects are estimated but may be confounded with two factor interactions.

Resolution IV is obtained for a 2^{4-1} design in which the generator is I = 1234. In this case main effects are estimated without confusion with two-factor interactions, whereas two factor interactions can be confounded with other two-factor interactions.

Resolution V is obtained with 2^{5-1} designs in which the generator is I = 12345. In this case main effects are estimated unconfounded by three-factor (or less) interactions, two-factor interactions effects are unconfounded by other two-factor interactions, but two-factor interactions may be confounded with three-factor interactions.

Higher resolutions (VI, VII, etc.) can be obtained for fractional designs based on 6 or more two-level factors, yet they are useless, since the expanded experimentation has no practical benefit in most cases.

Fractional designs with resolutions from III to V are thus the most important, with those having Resolutions III and IV being used only, or mainly, for screening.

Generally speaking, a fractional design is adopted when the following hypotheses are verified:

1) interactions between three or more factors are negligible;
2) interactions between two small effects are small;
3) interactions between a big and a small effect are usually small;
4) interactions between two big effects are likely big.

Plackett-Burman designs

Plackett-Burman designs were introduced in 1946 by English statisticians R.L. Plackett and J.P. Burman. Their main goal was to investigate the dependence of some measured quantity on a certain number of factors, each taking a certain number of levels, in such a way to minimize the variance of the estimates of these dependencies using a limited number of experiments.

The Plackett-Burman designs are very efficient for large numbers of factors, yielding a number of runs which is just one unity higher than the number of factors and, nonetheless, enabling a Resolution III (main effects confounded with binary interaction effects).

As shown in the next table, given a certain number of two-level factors, the difference with the number of experiments that would be required by a full factorial design is huge, when factors are more than 10:

Factors	11	15	19	23	27
Full factorial	2048	32768	524288	8.39×10^6	1.34×10^8
Plackett-Burman	12	16	20	24	28

A key aspect of Plackett-Burman designs is represented by the corresponding matrix, usually adopted for numbers of runs that are multiple of 4.

As an example, the matrix related to 11 factors, i.e., 12 runs, is the following:

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
1	+	+	−	+	+	+	−	−	−	+	−
2	−	+	+	−	+	+	+	−	−	−	+
3	+	−	+	+	−	+	+	+	−	−	−
4	−	+	−	+	+	−	+	+	+	−	−
5	−	−	+	−	+	+	−	+	+	+	−
6	−	−	−	+	−	+	+	−	+	+	+
7	+	−	−	−	+	−	+	+	−	+	+
8	+	+	−	−	−	+	−	+	+	−	+
9	+	+	+	−	−	−	+	−	+	+	−
10	−	+	+	+	−	−	−	+	−	+	+
11	+	−	+	+	+	−	−	−	+	−	+
12	−	−	−	−	−	−	−	−	−	−	−

Notably, whatever couple of factors X_i,X_j is considered, three equal combinations of their levels are found.

Moreover, 6 «+» and 6 «-» values of factors can be found in each column.

A more focused inspection of the matrix shows that the whole design can be generated starting from the first row. To build the second row, the last element of the first row is transformed into the first element of the second row, then the remaining elements of the first row are copied to the second row, shifted by one position to the right. The procedure is adopted down to the 11th row. The last row, *i.e.*, the 12th one, contains only "- " signs.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
1	+	+	−	+	+	+	−	−	−	+	−
2	−	+	+	−	+	+	+	−	−	−	+
3	+	−	+	+	−	+	+	+	−	−	−
4	−	+	−	+	+	−	+	+	+	−	−
5	−	−	+	−	+	+	−	+	+	+	−
6	−	−	−	+	−	+	+	−	+	+	+
7	+	−	−	−	+	−	+	+	−	+	+
8	+	+	−	−	−	+	−	+	+	−	+
9	+	+	+	−	−	−	+	−	+	+	−
10	−	+	+	+	−	−	−	+	−	+	+
11	+	−	+	+	+	−	−	−	+	−	+
12	−	−	−	−	−	−	−	−	−	−	−

Other Plackett–Burman matrices, for N corresponding to further multiples of 4, can be built using the same strategy but starting from the first rows reported in this table:

$N=8$	+ − − + − + +
$N=12$	+ + − + + + − − − + −
$N=16$	+ + + + − + − + + − − + − − −
$N=20$	+ + − − + + + + − + − + − − − − + + −
$N=24$	+ + + + + − + − + + − − + + − − + − + − − − −

An example of Plackett-Burman design

A company producing brake pads selected 11 variables as having a possible effect on the quality of the final product.
As a first screening, they were interested in sorting out which of these variables actually had an effect (or, better, in removing those variables that had no significant effect).
The selected variables and the levels under study are described in the following table:

	Factor	Level −	Level +
1	Resin type	Slow	Fast
2	Press type	Old	New
3	Press time	Short	Long
4	Press pressure	Low	High
5	Press temperature	Low	High
6	Oven temperature	Low	High
7	Oven time	Short	Long
8	*Scorching time	Short	Long
9	*Scorching temperature	Low	High
10	Pressure at high temperature	Low	High
11	Pressure at low temperature	Low	High

*The scorching process consists in heating the brake pad friction material to 600° – 700°C to reduce the time it takes for the pad and disc to adapt to each other.

Notably, some of the variables in the table (e.g., resin type, press type) can be described by a qualitative label, not a number. Though a numerical label can be applied to such qualitative variables, there is no correspondence at all with a real numerical value.

In the case of quantitative variables, the '-1' level is usually assigned to the lower level and the '+1' to the higher level; in the case of qualitative variables the '-1' and '+1' levels are arbitrarily assigned.

The following experimental matrix is obtained by including the response as the last column (pad compressibility):

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	Y
1	+	+	−	+	+	+	−	−	−	+	−	163
2	−	+	+	−	+	+	+	−	−	−	+	121
3	+	−	+	+	−	+	+	+	−	−	−	152
4	−	+	−	+	+	−	+	+	+	−	−	100
5	−	−	+	−	+	+	−	+	+	+	−	93
6	−	−	−	+	−	+	+	−	+	+	+	173
7	+	−	−	−	+	−	+	+	−	+	+	133
8	+	+	−	−	−	+	−	+	+	−	+	131
9	+	+	+	−	−	−	+	−	+	+	−	157
10	−	+	+	+	−	−	−	+	−	+	+	157
11	+	−	+	+	+	−	−	−	+	−	+	101
12	−	−	−	−	−	−	−	−	−	−	−	236

Since each column has six '-' and six '+' signs, each variable will have one half of the experiments performed at the low level and one half of the experiments performed at the high level.

As in the Factorial Design, the effect of each variable will be easily computed by calculating the algebraic sum of the responses, each with the appropriate sign.
This means that the effect of each variable will be derived from the comparison of the responses of the six experiments performed at the high level and the responses of the six experiments performed at the low level.
12 coefficients must be estimated for the model, and this is the reason why 12 experiments are required.

If the model equation is written as $y = b_0 + b_1x_1 + b_2x_2 + ... + b_{11}x_{11}$

b_0 is obtained as the sum of responses divided by 12: 1717 / 12 = 143.1.

In the case of b_1 the calculation is (see the table on the right):

(163-121+152-100-93-173+133+131+157-157+101-236)/12 = -3.6.

The same approach can be used to obtain all the other coefficients.

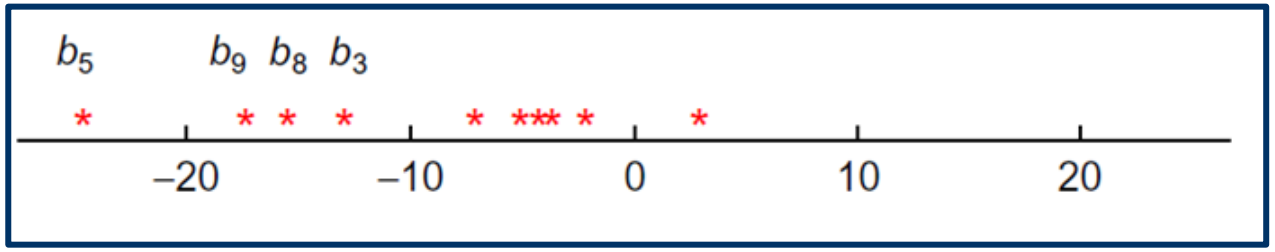
	X_1	Y
1	+	163
2	–	121
3	+	152
4	–	100
5	–	93
6	–	173
7	+	133
8	+	131
9	+	157
10	–	157
11	+	101
12	–	236

The coefficients of the model are reported in the table on the right:

As in all cases in which no estimation of experimental variability is made, it is not possible to have a statistical estimation of the significance of the coefficients.

An easy and visual way to sort out the 'relevant' variables is drawing a scatter plot of the coefficients and look for those variables whose coefficients lie far away from the bulk of the coefficients:

	$b_0 = 143.1$
Resin type	$b_1 = -3.6$
Press type	$b_2 = -4.9$
Press time	$b_3 = -12.9$
Press pressure	$b_4 = -2.1$
Press temperature	$b_5 = -24.6$
Oven temperature	$b_6 = -4.2$
Oven time	$b_7 = -3.8$
Scorching time	$b_8 = -15.4$
Scorching temperature	$b_9 = -17.2$
Pressure at high temperature	$b_{10} = +2.9$
Pressure at low temperature	$b_{11} = -7.1$



It can be seen that four coefficients have particularly high absolute values: b_3, b_5, b_8 and b_9 (press time, press temperature, scorching time and scorching temperature).
As their coefficients are negative and the response has to be minimized, it can be concluded that for both steps of the process (press and scorching) longer times and higher temperatures improve the quality of the product.

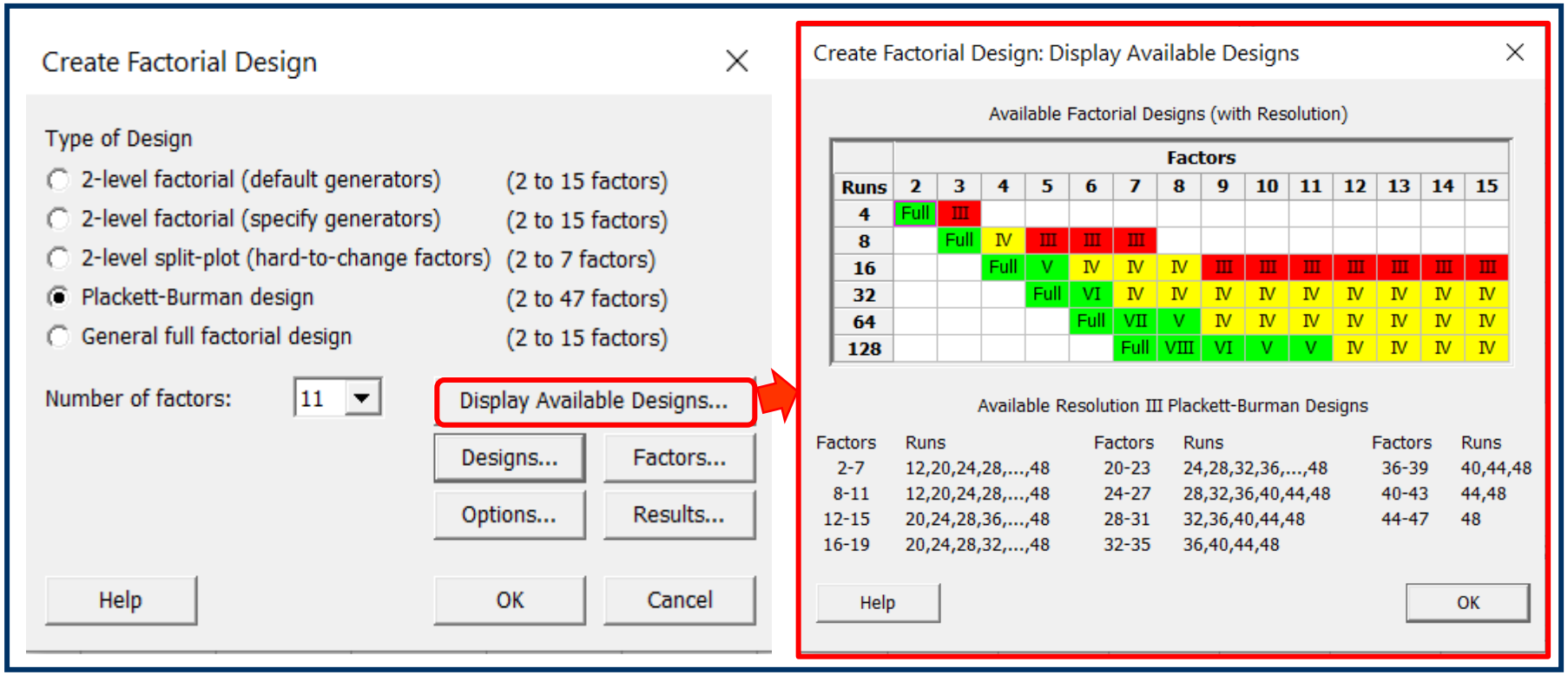
More detailed results can now be obtained by preparing a new experimental design (*e.g.*, a Factorial Design) with these four variables only, all of them having a range moved toward longer times and higher temperatures.

Use of Minitab for calculations on a Plackett-Burman design

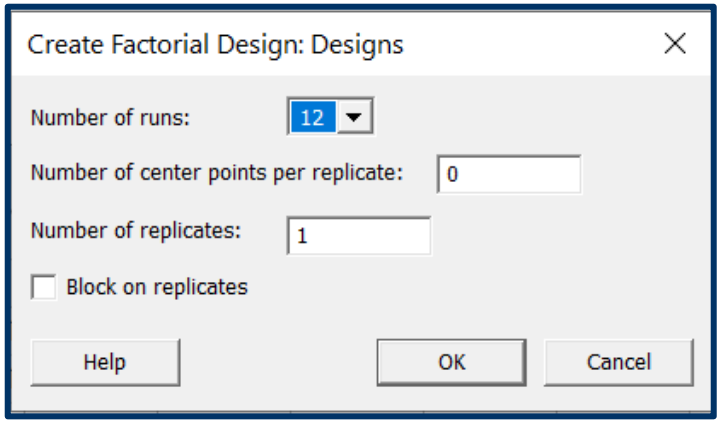
The Minitab software includes also an option for the Plackett-Burman design.

The option is accessed from the same path described before for factorial design: Stat > DOE > Factorial Design > Create Factorial Design….

When the «Display Available Designs…» window is opened, a table describing all available Plackett-Burman Designs, along with full or fractional factorial designs, with indication of the respective resolutions for the latter, is shown:



The specific conditions related to the adopted Plackett-Burman design can be indicated by accessing the «Designs» window in the «Create Factorial Design» window:



In the present case, 12 runs are indicated, since 11 factors are considered. Only one replicate is assumed for each run.

When the OK command is given, a Worksheet with the coded values of factors to be used in the 12 runs, along with their randomized order, is provided by Minitab and response values can be added in the last column once all runs have been performed:

Worksheet 2 ***

↓	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	C14	C15	C16
	StdOrder	RunOrder	PtType	Blocks	A	B	C	D	E	F	G	H	J	K	L	Response
1	9	1	1	1	-1	-1	-1	1	1	1	-1	1	1	-1	1	157
2	7	2	1	1	-1	1	1	1	-1	1	1	-1	1	-1	-1	133
3	3	3	1	1	-1	1	1	-1	1	-1	-1	-1	1	1	1	152
4	5	4	1	1	1	1	-1	1	1	-1	1	-1	-1	-1	1	93
5	8	5	1	1	-1	-1	1	1	1	-1	1	1	-1	1	-1	131
6	11	6	1	1	-1	1	-1	-1	-1	1	1	1	-1	1	1	101
7	10	7	1	1	1	-1	-1	-1	1	1	1	-1	1	1	-1	157
8	4	8	1	1	1	-1	1	1	-1	1	-1	-1	-1	1	1	100
9	2	9	1	1	1	1	-1	1	-1	-1	-1	1	1	1	-1	121
10	12	10	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	236
11	1	11	1	1	1	-1	1	-1	-1	-1	1	1	1	-1	1	163
12	6	12	1	1	1	1	1	-1	1	1	-1	1	-1	-1	-1	173

It is worth noting that Minitab uses a specific table of coded values for the 12 runs, different from that reported before, yet the basic constraints are respected.

Just to obtain an example of the program output, response data used before were introduced in the Minitab worksheet as if they were obtained with combinations of factors adopted before.

Once all data were introduced, the design analysis was made through the already described pathway: Stat > DoE > Factorial Design > Analyze Factorial Design…

A table with coded coefficients, i.e., coefficients of the regression model obtained using coded values of factors, was reported in the program's output.

The regression equation in uncoded units is also reported, yet in this case it included the same coefficients, since the coding rule was not provided to the software.

Coded Coefficients

Term	Effect	Coef	SE Coef	T-Value	P-Value	VIF
Constant		143.1	*	*	*	
A	-17.167	-8.583	*	*	*	1.00
B	-28.50	-14.25	*	*	*	1.00
C	-2.167	-1.083	*	*	*	1.00
D	-41.17	-20.58	*	*	*	1.00
E	1.5000	0.7500	*	*	*	1.00
F	-12.500	-6.250	*	*	*	1.00
G	-26.83	-13.42	*	*	*	1.00
H	-4.167	-2.083	*	*	*	1.00
J	8.167	4.083	*	*	*	1.00
K	-32.17	-16.08	*	*	*	1.00
L	-30.83	-15.42	*	*	*	1.00

Regression Equation in Uncoded Units

Response = 143.1 - 8.583 A - 14.25 B - 1.083 C - 20.58 D + 0.7500 E - 6.250 F - 13.42 G - 2.083 H + 4.083 J - 16.08 K - 15.42 L